AUTOMATIC ERROR ANALYSIS IN FINITE DIGITAL COMPUTATIONS, USING RANGE ARITHMETIC

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525

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[redacted]

Submitted in partial fulfillment
for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

UNITED STATES NAVAL POSTGRADUATE SCHOOL

May 1966

ABSTRACT

The nature of generated machine error in finite digital calculations is discussed. The arithmetic of range numbers is developed, and examples are given demonstrating the use of range arithmetic as a tool for automatic error analysis. A computer program is developed, utilizing the TYPE OTHER feature of FORTRAN-63 in conjunction with the CDC-1604 digital computer, which enables the user to perform automatic error analysis during computation, and a number of programs are presented using this feature.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

1. Introduction.

The utilization of high-speed digital computers to perform lengthy and complex mathematical computations is widespread. Unfortunately, the user often assumes that the computer output will necessarily be accurate if his input data is accurate and his program logic is sound. This need not be the case, and such an assumption may well lead the user to incorrect or highly inaccurate results. The difficulty arises because of the very nature of digital computation.

A digital computer has a finite memory, and cannot carry within it infinite precision numbers, or perform infinitely accurate calculations. Each number stored in a digital computer must, of necessity, contain a finite number of digits. As a result, during each step of computation, the computer automatically truncates or rounds off numbers to conform with its memory storage requirements. Each arithmetic operation introduces an error in precision due to such truncation, which may accumulate with other such errors to produce sizeable inaccuracies in results. It is also possible that the errors may cancel and give a fairly accurate result; but, in either case, the user will have no knowledge of whether or not any error had accumulated in computation.

It would be desirable to eliminate precision errors entirely, but hardware limitations make it impossible to do so. It is, however, possible to estimate and control the errors arising in finite digital computations, by appropriate choice of an algorithm. One method is

called <u>range arithmetic</u>, or <u>interval arithmetic</u> [1, 2]. This method enables the generated error to be analysed automatically within the computer, during computation.

The technique of range arithmetic replaces any real number of infinite precision by two numbers of finite precision, one of which is a lowerbound of the real number, and the other is a corresponding upperbound of the real number. We can then be certain that the real number lies in the range of the two finite precision numbers.

In this thesis, we will discuss the arithmetic of range numbers, the application of range arithmetic to digital computation, and present a computer program designed for use on the CDC-1604 computer in conjunction with FORTRAN-63, which will provide a means of automatic error analysis in digital computation.

2. The Arithmetic of Range Numbers.

In this section, we will discuss the properties of range numbers, and the arithmetic operations pertaining to them.

Let x be any real number. Associated with this real number is a range number X, which has the following properties:

(1) X is a closed bounded interval;

(2) X contains x.

Then define the range number X, associated with the real number x, to be the closed bounded interval

$$X = [xL, xU],$$

where xL and xU are defined by the transformations L and U as follows:

$$x \xrightarrow{L} xL$$

$$x \xrightarrow{U} xU$$

We may then write

$$x \xrightarrow{L,\ U} X = [xL, xU] \quad .$$

The transformations L and U are determined by the precision allowed in computation. If, for example, we would like to find the range number associated with $e = 2.71828 \ldots\ldots$, and we are allowed only two significant figures, then

$$X = [2.7, 2.8] \ .$$

On the other hand, if allowed five significant figures, then

$$X = [2.7182, 2.7183] \ .$$

For integers (known with exact precision), the range number will have the same upper and lower bound, equal to the integer itself. Then, if $x = 3$, $X = [3, 3]$.

We can see, then, that the range number X will represent the smallest closed interval containing x, for a given set of transformations L and U.

The arithmetic operations associated with range numbers can be defined in the following manner. Let (op) represent an arithmetic operation, and consider two range numbers

$$X = [xL, xU] ,$$

and

$$Y = [yL, yU] .$$

Then,

$$Z = X \text{ (op) } Y = [zL, zU] ,$$

where

$$zL = \min[xL \text{ (op) } yL, xL \text{ (op) } yU, xU \text{ (op) } yL, xU \text{ (op) } yU]$$

$$zU = \max[xL \text{ (op) } yL, xL \text{ (op) } yU, xU \text{ (op) } yL, xU \text{ (op) } yU] .$$

It can be seen that the real number $z = x \text{ (op) } y$ will be in the interval Z, and that Z represents the smallest interval that contains z.

Now let

(+) - denote range addition

(-) - denote range subtraction

(*) - denote range multiplication

(/) - denote range division

From the definition above, it can be seen that

(1) $Z = X\,(+)\,Y = [\,xL + yL\,,\ xU + yU\,]$

(2) $-X = [\,-xU\,,\ -xL\,]$

(3) $Z = X\,(-)\,Y = X\,(+)\,(-Y) = [\,xL - yU\,,\ xU - yL\,]$

(4) $X = Y$ if and only if

$xL = yL$, and $xU = yU$

(5) $Z = X\,(*)\,Y$, where

$zL = \min\,[\,xL * yL\,,\ xL * yU\,,\ xU * yL\,,\ xU * yU\,]$

$zU = \max\,[\,xL * yL\,,\ xL * yU\,,\ xU * yL\,,\ xU * yU\,]$

The values of zL and zU can also be obtained using a table of sign discriminations.

Using the table on page 14, we need only calculate one product for each end point, except in sign discrimination 5.

(6) $Z = X\,(/)\,Y$, where

$zL = \min\,[\,xL / yL\,,\ xL / yU\,,\ xU / yL\,,\ xU / yU\,]$

$zU = \max\,[\,xL / yL\,,\ xL / yU\,,\ xU / yL\,,\ xU / yU\,]$

Note, that in the case in which Y brackets zero, i. e., $Y = [\,-1\,,\ 3\,]$, range division is undefined. This is equivalent to forbidden division by zero in real arithmetic.

	xL	xU	yL	yU	Values of zL and zU
1	+	+	+	+	zL = xL*yL , zU = xU*yU
2	+	+	-	+	zL = xU*yL , zU = xU*yU
3	+	+	-	-	zL = xU*yL , zU = xL*yU
4	-	+	+	+	zL = xL*yU , zU = xU*yU
5	-	+	-	+	zL = min [xU*yL , xL*yU], zU = max [xL*yL , xU*yU]
6	-	+	-	-	zL = xU*yL , zU = xL*yL
7	-	-	+	+	zL = xL*yU , zU = xU*yL
8	-	-	-	+	zL = xL*yU , zU = xL*yL
9	-	-	-	-	zL = xU*yU , zU = xL*yL

TABLE OF SIGN DISCRIMINATIONS

FIGURE 1

We now will give some illustrative examples of computations using range arithmetic.

$$[a, b]\,(+)\,[c, c] = [a+c,\ b+c]$$

$$[a, b]\,(*)\,[c, c] = [ca,\ cb] \qquad \text{for } c \geq 0$$

$$[a, b]\,(*)\,[c, c] = [cb,\ ca] \qquad \text{for } c \leq 0$$

$$[a, b]\,(-)\,[a, b] = [a-b,\ b-a]$$

$$[a, b]\,(/)\,[a, b] = [a/b,\ b/a] \qquad \text{for } a > 0$$

$$[1, 2]\,(*)\,[-4, -3] = [-8, -3]$$

3. Range Arithmetic and Automatic Error Analysis.

In finite digital computation, errors are generated in the process of round-off, truncation, and normalization. One has no method of determining the seriousness of such errors unless some sort of error analysis is performed. A method is necessary that can evaluate the error generated at each step in the computation, and also keep track of the accumulated error. The techniques of range arithmetic are suited to this purpose. If, in addition to doing real arithmetic, we parallel the computation with range arithmetic, the range numbers calculated will keep track of the error, and the final result will be the smallest closed interval containing the "true", infinite precision result.

Error analysis via range arithmetic not only furnishes information as to the size of machine-generated errors, but it also can be utilized to evaluate the accuracy of alternative computational schemes. Indeed, the error generated in one type of computation may be quite different from that obtained using another method of computation. A few examples will illustrate this point.

Example 1.

Consider the problem of finding the roots of the equation

$$x^2 + 100,000x + 100 = 0 \ .$$

We will concern ourselves with finding the larger root of this equation, and we will assume that the computer can carry six significant

figures. We will first calculate the result using ordinary arithmetic and the quadratic formula, and then use range arithmetic in conjunction with the quadratic formula. The notation will be that which is equivalent to normalized floating-point operations on a computer. For example, the number 100,000 will be denoted by .100000E6, where E6 denotes the power of ten multiplying .100000.

The equation

$$ax^2 + bx + c = 0$$

has, as its larger root,

$$x = (-b + \sqrt{b^2 - 4ac})/2a \ .$$

We then calculate our solution:

$$x = \frac{-.100000\text{E}6 + \sqrt{.100000\text{E}11 - .400000\text{E}3}}{.2\text{E}1}$$

$$= \frac{-.100000\text{E}6 + \sqrt{.100000\text{E}11 - .000000\text{E}11}}{.2\text{E}1}$$

$$= \frac{-.100000\text{E}6 + .100000\text{E}6}{.2\text{E}1}$$

$$= .000000\text{E}00$$

or

$$x = 0$$

Now, in order to do the same problem in range arithmetic, we assume some uncertainty in the coefficients of our original equation. We then replace the original coefficients by range numbers reflecting the uncertainty. For example, we will replace 100,000 by [99999.9, 100,001].

The equation to be solved now is

$$[.999999E0,\ .100001E1]\,x^2 + [.999999E5,\ .100001E6]\,x$$

$$+ [.999999E2,\ .100001E3] = 0\ .$$

Now,

$$b^2 = [.999999E5,\ .100001E6]^2$$

$$= [.99999E10,\ .100003E11]$$

$$ac = [.999999E0,\ .100001E1] \times [.999999E2,\ .100001E3]$$

$$= [.999998E2,\ .100003E3]$$

$$4ac = [.399999E3,\ .400012E3]$$

$$b^2 - 4ac = [.999998E10, .100003E11] - [.399999E3,\ .400012E3]$$

$$= [.999997E10, .100003E11]$$

$$\sqrt{b^2 - 4ac} = [.999998E5,\ .100002E6]$$

$$-b + \sqrt{b^2 - 4ac} = [.999998E5,\ .100002E6] - [.999999E5,\ .100001E6]$$

$$= [-.120000E4,\ .210000E4]$$

and

$$x = \frac{[-.120000E4,\ .210000E4]}{[.399999E3,\ .400012E3]}$$

$$= [-.299994E1,\ .525002E1]$$

By using range arithmetic, we have kept accurate error bounds and have found that our root is in the interval [- 2.9, 5.3], or [- .299994E1, .525002E1]. Our result tells us that the true value could be anywhere within the calculated interval. However, the interval

is wide, and could contain the value zero, which was our answer using real arithmetic. We would like to tighten the interval, so that we may be more confident of the computer results. The next example will illustrate how, with a little knowledge of sources of error generated in finite calculations, we can obtain more satisfactory results.

Example 2.

Consider the formula for the larger root of a quadratic equation,

$$x = (-b + \sqrt{b^2 - 4ac}) / 2a$$

and divide numerator and denominator by b so that

$$x = (-1 + \sqrt{1 - s}) b / 2a$$

where

$$s = 4ac / b^2 .$$

Expanding the radical in a power series, we obtain

$$\sqrt{1 - s} = 1 - s/2 - s^2/8 - s^3/16 - \ldots\ldots\ldots$$

The equation for x becomes

$$x = - (s/2 + s^2/8 + s^3/16 + \ldots\ldots) b/2a .$$

Now, for the equation

$$x^2 + 100,000x + 100 = 0$$

we get

$$s = \frac{(.400000E1)\ (.100000E1)\ (.100000E3)}{(.100000E6)^2}$$

$$= .400000E-7$$

$$s/2 = .200000E-7$$

$$s^2/8 = .200000E-15$$

$$b^2/2a = (.100000E6)/(.200000E1)(.100000E1)$$

$$= .500000E5$$

Then

$$x = -(.500000E5)(.200000E-7 + .200000E-15)$$

$$= -(.500000E5)(.200000E-7)$$

$$= -.100000E-2$$

or,

$$x = -.001$$

We now solve the same problem using range arithmetic, with the modified quadratic formula.

Our equation is

$$[.999999E0,\ .100001E1]\,x^2 + [.999999E5,\ .100001E6]\,x$$
$$+ [.999999E2,\ .100001E3] = 0 .$$

In example 2, we found that

$$b^2 = [.999998E10,\ .100003E11]$$

$$4ac = [.399999E3,\ .400012E3]$$

therefore,

$$s = \frac{[.399999E3,\ .400012E3]}{[.999998E10,\ .100003E11]}$$

$$= [.399987E-7,\ .400013E-7]$$

$$s/2 = [.399987E-7,\ .400013E-7]/[.200000E1,\ .200000E1]$$

$$= [.199994E-7,\ .200007E-7]$$

$$s^2 = [.159987E\text{-}14,\ .160009E\text{-}14]$$

$$s^2/8 = [.199983E\text{-}15,\ .200012E\text{-}15]$$

$$s/2 + s^2/8 = [.199994E\text{-}7,\ .200008E\text{-}7]$$

$$b/a = [.999999E5,\ .100001E6]/[.999999E0,\ .100001E1]$$

$$= [.999989E5,\ .100002E6]$$

$$b/2a = [.499994E5,\ .500005E5]$$

and

$$x = -[.499994E5,\ .500005E5]\,[.199994E\text{-}7,\ .200008E\text{-}7]$$

$$= [-.100005E\text{-}2,\ -.999958E\text{-}3]$$

or,

$$x = [-.00100005,\ -.000999958]\ .$$

Thus, we can see that the method in example 2 generates far less error than the straight application of the quadratic formula. Indeed, the previous examples illustrate the use of range arithmetic in evaluating a given computational technique.

4. QRANGE7 - Program Description and Usage.

The TYPE OTHER capability in the FORTRAN-63 compiler for the CDC-1604 computer allows the relatively simple implementation of range arithmetic for the user. Using the QRANGE7 package, the programmer need only declare all floating-point variables to be TYPE RANGE7(3), and follow the usual rules of FORTRAN programming. The QRANGE7 package supplies the necessary subroutines for range arithmetic computations.

Range arithmetic calculations are made carrying a triple of numbers (xF, xL, xU), rather than a double as stated previously. The first number in the triple is the ordinary floating-point result, as calculated using standard floating-point arithmetic. The remaining numbers in the triple are the lower and upper range numbers, respectively. Note that $xL \leq xF \leq xU$. In any range calculation, the "real" part of the triple is computed last and remains in the accumulator, so that transfers and comparisons (such as the IF statement) will follow the same logical branches as in ordinary floating-point arithmetic.

Standard floating-point arithmetic cannot be used on either lower or upper range numbers since, in floating-point calculations, round-off and normalization are done automatically, and would reduce the accuracy of the range interval. In QRANGE7, the arithmetic performed on range numbers is un-normalized, and separate subroutines are provided which truncate the lower range numbers and round up by one bit in the

least significant position, or truncate, the upper range numbers as required. The un-normalized arithmetic is performed by unpacking the operands, putting them in un-normalized form, doing the appropriate arithmetic, re-normalizing and repacking the result.

Input/output may be accomplished in one of two ways. The programmer may use EQUIVALENCE statements to transfer input / output of range numbers as real numbers, where each range number is equivalent to three real numbers, or he may use the Input / Output for Multi-Word TYPE OTHER package [3] and transfer the range numbers directly in and out of the computer memory. The following examples will illustrate the use of these methods, and the use of QRANGE7.

A.

```
      TYPE RANGE7(3)  A7, B7, X7
      DIMENSION  A7(3), AR(9), B7(3), BR(9), X7(3, 3), XR(27)
      EQUIVALENCE  (A7, AR), (B7, BR), (X7, XR)
      READ  10, AR, BR
   10 FORMAT(8F10. 0)
      DO  20  I=1, 3
      DO  20  J=1, 3
   20 X7(J, I) = A7(I)*B7(J)
      DO  30  I=1, 25, 3
   30 PRINT  40, XR(I), XR(I+1), XR(I+2), XR(I), XR(I+1),  XR(I+2)
   40 FORMAT(//15X, 3(E20. 10, 10X)/15X, 3(020, 10X))
      END
```

In this example, we have assumed that the programmer reads in the three parts of the range numbers using real floating-point numbers that are simultaneously stored as the three parts of the range numbers by use of the EQUIVALENCE statement. The next example will illustrate the same program, but using the INPUT / OUTPUT for MULTI-WORD TYPE OTHER package [3].

B.

```
      TYPE  RANGE7(3)  A7, B7, X7
      DIMENSION  A7(3), B7(3), X7(3, 3)
      CALL  READ7
      READ  10, A7, B7
   10 FORMAT(8F10.0)
      DO  20  I=1, 3
      DO  20  J=1, 3
      X7(J, I) = A7(I)*B7(J)
      CALL  PRINT7
   20 PRINT  30,  X7(J, I)
   30 FORMAT(//15X, 3(E20.10, 10X) / 15X, 3(020, 10X))
      END
```

5. Matrix Inversion Using Range Arithmetic.

In order to test QRANGE7, and evaluate the errors generated in a particular program, a number of matrix inversions were run. In one case, the matrix to be inverted was generated randomly, using the RANF(-1) library function of FORTRAN-63. About 15 programs were run in this manner. In the second case, the input matrix was specified, and the individual elements of the matrix were assumed to be inaccurate. Thus, they were inputed as range numbers. Both cases used the MATINV2 subroutine provided by the U. S. Naval Postgraduate School computer facility, in conjunction with QRANGE7. Two range subroutines had to be used: ABS7, which takes the absolute value of a range number; and Q0Q06700, which complements the range accumulator [5]. (A listing of these subroutines appears in Appendix II.) The coding for both cases was as follows:

Case 1. Randomly Generated Matrices

```
      PROGRAM RANMAT
      TYPE RANGE7(3) XMAT,XINV
      DIMENSION XMAT(5,5),XINV(5,5),XJUNK(2000)
      READ 2000,N
      DO 90 I=1,N
   90 XJUNK(I) = RANF(-1)
      DO 10 I=1,5
```

```
      DO 10 J=1,5
      B(I,J) = RANF(-1)*1000
   10 XMAT(I,J) = B(I,J)
      CALL PRINT7
      DO 20 J=1,5
      DO 20 I=1,5
   20 PRINT 2001,XMAT(I,J),XMAT(I,J)
      CALL MATINV2(XMAT,XINV,5,5)
      CALL PRINT7
      DO 30 J=1,5
      DO 30 I=1,5
   30 PRINT 2001,XINV(I,J),XINV(I,J)
 2000 FORMAT(F10.0)
 2001 FORMAT(//15X,3(E20.10,10X)/15X,3(020,10X))
      END
```

Case 2. Input Matrix Containing Range Elements

```
      PROGRAM MMATRIX
      TYPE RANGE7(3) A,B
      DIMENSION A(3,3),B(3,3)
      CALL READ7
      DO 10 J=1,3
      DO 10 I=1,3
   10 READ 2000,A(I,J)
```

```
      CALL  PRINT7
      DO  20  J=1,3
      DO  20  I=1,3
20    PRINT  2001 , A(I, J) , A(I, J)
      CALL  MATINV2(A, B, 3, 3)
      CALL PRINT7
      DO  30  J=1,3
      DO  30  I=1,3
30    PRINT  2001 , B(I, J) , B(I, J)
2000  FORMAT(3F10.0)
2001  FORMAT(//15X , 3(E20.10, 10X) / 15X, 3(020, 10X))
      END
```

The results obtained using random matrices were highly accurate. For 5 x 5 matrices, seven decimal places of accuracy were obtained, and the range of error was small. It might be noted that the conversion of the real random matrix to a matrix of range numbers introduced no errors, and hence the resulting range intervals on the inverse elements were solely a consequence of digital computation. On the other hand, the error generated in the second case was quite large, and decimal accuracy varied from zero to one. In fact, large errors were generated for input coefficients with ranges of approximately 1.0E-2. It was found, in both cases, that the error generated increased with increasing matrix dimension. This is to be expected, since the number of

calculations done in inverting the matrix is increased. For example, a 12 x 12 random matrix was inverted, and it was found that only four to five significant figures could be expected, vice the seven decimal accuracy in the 5 x 5 case.

One my circumvent the inaccuracies obtained when the input elements are range numbers, by the following device [4]. Suppose we wish to invert X, whose elements are range numbers. Take X_0 to be the matrix of mid-points of the elements of X, so that $X = X_0 + E$. The elements of E are range numbers of the form $[-e, e]$. Obtain X_0^{-1} using range arithmetic, and let

$$E' = [I - X_0^{-1}E + (X_0^{-1}E)^2 - \dots\dots\dots] X_0^{-1} (-EX_0^{-1}).$$

If the elements of E are small compared to X_0^{-1}, then the series will converge, and one can write

$$X^{-1} = X_0 + E'.$$

The results of a typical run for an inversion of a randomly generated 5 x 5 matrix are shown on the following page.

Element	Real Part	Lower	Upper
(1, 1)	4.6396189281E01	Same as Real	Same as Real
(1, 2)	8.6916184500E02		
(1, 3)	3.6039163981E02		
(1, 4)	5.8772778565E02		
(1, 5)	9.4693749440E02		
(2, 1)	5.5609401546E02		
(2, 2)	8.9089112203E02		
(2, 3)	4.0143081011E02		
(2, 4)	4.2098029158E02		
(2, 5)	6.1093177836E02		
(3, 1)	9.9636585643E02		
(3, 2)	1.6340010236E02		
(3, 3)	4.6658037215E02		
(3, 4)	5.0479887773E02		
(3, 5)	2.0507283101E02		
(4, 1)	2.7500284988E02		
(4, 2)	4.8510492363E02		
(4, 3)	2.4488146363E02		
(4, 4)	4.1884968178E02		
(4, 5)	1.9965179004E02		
(5, 1)	8.7792113349E02		
(5, 2)	2.3254672355E02		
(5, 3)	3.5002222285E00		
(5, 4)	3.2092384145E02		
(5, 5)	6.4308479314E02		

X - MATRIX

TABLE 2

We will present the real part, and the least significant digits of the lower and upper range numbers of X - INVERSE. The exponents will not be repeated.

Element	Real Part		Lower	Upper
(1, 1)	-9.1422040	728E-04	854	607
(1, 2)	7.419425	7112E-04	6975	7249
(1, 3)	9.3418389	435E-05	070	864
(1, 4)	-5.779019	5190E-05	5709	4688
(1, 5)	6.2948717	498E-04	477	517
(2, 1)	-1.122627	041E-03	052	032
(2, 2)	1.6441862	954E-03	944	965
(2, 3)	-1.3201931	162E-03	165	159
(2, 4)	1.4714936	870E-03	865	875
(2, 5)	5.5237176	832E-05	720	995
(3, 1)	1.0620289	855E-03	838	873
(3, 2)	1.1383145	460E-03	437	478
(3, 3)	2.3018511	807E-03	800	812
(3, 4)	-3.6993050	965E-03	973	956
(3, 5)	-2.2307817	966E-03	966	962
(4, 1)	6.4802494	170E-04	040	338
(4, 2)	-3.0018154	973E-03	990	955
(4, 3)	-5.102677	3200E-05	3669	2679
(4, 4)	4.0549999	820E-03	811	825
(4, 5)	6.5487719	463E-04	433	477
(5, 1)	1.3248530	755E-03	742	766
(5, 2)	-1.1561055	861E-04	968	706
(5, 3)	3.6280024	837E-04	794	865
(5, 4)	-2.4566800	524E-03	531	520
(5, 5)	3.6100559	502E-04	481	523

X - INVERSE

FIGURE 3

6. Conclusions.

Range arithmetic is a powerful tool, not only for error estimation in computations of a scientific or engineering nature, but for the evaluation and comparison of alternative numerical algorithms as well. The QRANGE7 package provides the user with a simple means for such analysis. The drawback in the use of QRANGE7, at this point, is that computing time is increased. There are many points where the program could be increased in efficiency, notably in the Q1Q04770 and Q1Q05770 subroutines for range multiplication and range division. The author feels, however, that even with the increase in computing time, the payoff in the use of range arithmetic is so great that it more than outweighs the disadvantage.

A few function subprograms have been provided for use in conjunction with QRANGE7. They are used exactly as one would use the library subroutines in FORTRAN, except a "7" replaces the "F" at the end of the function name. One must remember to declare these functions TYPE RANGE7(3) before execution. Much work remains to be done in this area, and it is hoped that, eventually, a complete library of functions will be available for use with QRANGE7.

It might be noted, at this point, that there are errors generated in the conversion of decimal-to-binary and binary-to-decimal numbers during input/output, and these have not been taken into account. The author feels that these errors are negligible compared to the machine

generated error occurring during computation.

The results obtained using QRANGE7 indicated that the technique of range arithmetic can, indeed, be used to provide error information at each stage of computation. Unfortunately, the author did not have the time to try it out on some of the more common numerical algorithms, such as the Runge - Kutta method for solving differential equations, or the Newton - Raphson method for determining roots of n^{th} order polynomials. It is hoped that this will be done in the future.

In conclusion, it is felt that range arithmetic is of such value that it warrants inclusion in future modifications of FORTRAN or other algebraic compilers as a standard TYPE.

BIBLIOGRAPHY

1. Moore, R. Lockheed Missiles - Document, LMSD-48421, 1959.

2. Shudde, R. A Computational Pitfall and Automatic Error Analysis, Atomics International Applied Mathematics Technical Document (AMTD-133), May, 1962.

3. Shudde, R. Input / Output for Multi-Word TYPE OTHER, Program Description 15/J5-NPGS-OTHERI/O (FTN-63), U. S. Naval Postgraduate School, April, 1966.

4. Collins, G. Interval Arithmetic for Automatic Error Analysis, Data Systems Division International Business Machine Corporation, Report M & A-5, 1960.

5. FORTRAN-63 / Reference Manual, Control Data Corporation, June, 1964.

APPENDIX I

QRANGE7 PROGRAM LISTING [5]

Entry Points:

Q1Q00770, Q1Q01770, Q1Q02770, Q1Q03770,

Q1Q04770, Q1Q05770, Q1Q10770, Q1Q00710,

Q1Q01710, Q1Q02710, Q1Q03710, Q1Q04710,

Q1Q05710, Q1Q10710, Q1Q10170, Q1Q00700,

Q1Q01700, Q1Q02700, Q1Q03700, Q1Q04700,

Q1Q05700, Q1Q10700, Q1Q10070, UNPACK7,

RNGAD7, RNGMU7, RNGDI7, QUIRKL7, QUIRKU7,

REPACK7, ERROR777

```
         IDENT    QRANGE7
RNG7S777 BLOCK    4
         COMMON   ACC(3),B
A        BSS      4
Q        BSS      4
T        BSS      3
A.       BSS      3
Q.       BSS      3
TEMP0    BSS      1
TEMP1    BSS      1
TEMP2    BSS      1
TEMP3    BSS      1
TEMP4    BSS      1
MTEMPU   BSS      1
DTEMPU   BSS      1
CA       BSS      1
CB       BSS      1
EA       BSS      1
EB       BSS      1
EL       BSS      1
EM1      BSS      1
EM2      BSS      1
FM1      BSS      1
FM2      BSS      1
BIT      BSS      1
S        BSS      1
C        BSS      1
AS       BSS      3
AS.      BSS      3
SA       BSS      3
AI       BSS      1
IA       BSS      1
AUX      BSS      3
AUX.     BSS      3
SAUX     BSS      3
```

```
SAUX.    BSS      3
QAUX     BSS      3
AF       BSS      1
         ENTRY    Q1Q00770                 THIS PROGRAM LOADS RANGE          1Q0000
Q1Q00770 SLJ      **                       ACCUMULATOR.   THE LOWER          1Q0010
         LDA      *                        GOES TO ACC+1,THE UPPER TO        1Q0020
         ALS      +24                      ACC+2, AND THE REAL PART          1Q0030
         INA      -1                       GOES TO ACC.                      1Q0040
         SAU      *+1                                                        1Q0050
+        ENA    7 **                                                         1Q0060
         SAU      A1                                                         1Q0070
         INA      +1                                                         1Q0080
         SAU      A2                                                         1Q0090
         INA      +1                                                         1Q0100
         SAU      A3                                                         1Q0110
A2       LDA      **                       LOAD LOWER RANGE NUMBER           1Q0120
         STA      ACC+1                    LOWER TO ACC+1                    1Q0130
A3       LDA      **                       LOAD UPPER RANGE NUMBER           1Q0140
         STA      ACC+2                    UPPER TO ACC+2                    1Q0150
A1       LDA      **                       LOAD REAL PART                    1Q0160
         STA      ACC                      REAL TO ACC                       1Q0170
         SLJ      Q1Q00770                                                   1Q0180
         ENTRY    Q1Q01770                 LOADS RANGE ACCUMULATOR           2Q0000
Q1Q01770 SLJ      **                       WITH THE COMPLEMENT OF A          2Q0010
         LDA      *                        RANGE NUMBER.                     2Q0020
         ALS      +24                      NOTE, WHEN LOWER AND UPPER        2Q0030
         INA      -1                       ARE COMPLEMENTED,THEIR ORDER      2Q0040
         SAU      *+1                      IS REVERSED.                      2Q0050
+        ENA    7 **                                                         2Q0060
         SAU      B1                                                         2Q0070
         INA      +1                                                         2Q0080
         SAU      B2                                                         2Q0090
         INA      +1                                                         2Q0100
         SAU      B3                                                         2Q0110
B2       LAC      **                       LOAD LOWER COMPLEMENT             2Q0120
```

```
         STA      ACC+2                  STORE IN ACC+2 (NEW UPPER)          2Q0130
B3       LAC      **                     LOAD UPPER COMPLEMENT               2Q0140
         STA      ACC+1                  STORE IN ACC+1 (NEW LOWER)          2Q0150
B1       LAC      **                     LOAD REAL COMPLEMENT                2Q0160
         STA      ACC                    STORE IN ACC                        2Q0170
         SLJ      Q1Q01770                                                   2Q0180
         ENTRY    Q1Q02770               ROUTINE FOR RANGE (+) RANGE,        3Q0000
Q1Q02770 SLJ      **                     THAT IS  A7(+)B7. A7 IS IN          3Q0010
         LDA      *                      RANGE ACCUMULATOR.                  3Q0020
         ALS      +24                                                        3Q0030
         INA      -1                                                         3Q0040
         SAU      *+1                                                        3Q0050
+        ENA    7 **                                                         3Q0060
         SAL      C1                                                         3Q0070
         INA      +1                                                         3Q0080
         SAU      C2                                                         3Q0090
         INA      +1                                                         3Q0100
         SAU      C3                                                         3Q0110
         LDA      ACC+1                  LOADS ACC+1 (A7 LOWER)              3Q0120
         STA      B                      STORES IN B                         3Q0130
C2       LDA      **                     LOADS B7 LOWER                      3Q0140
         RTJ      UNPACK7                UNPACKS BOTH LOWER RANGE            3Q0150
+        RTJ      RNGAD7                 NUMBERS,ADDS THEM                   3Q0160
+        RTJ      QUIRKL7                TRUNCATES UNPACKED RESULT, AND      3Q0170
+        RTJ      REPACK7                RENORMALIZES RESULT                 3Q0180
+        STA      ACC+1                  REPACKS AND STORES IN ACC+1         3Q0190
         LDA      ACC+2                  LOADS ACC+2 (A7 UPPER)              3Q0200
         STA      B                      STORES IN B                         3Q0210
C3       LDA      **                     LOADS B7 UPPER                      3Q0220
         RTJ      UNPACK7                UNPACKS A7 UPPER AND B7 UPPER       3Q0230
+        RTJ      RNGAD7                 ADDS THEM                           3Q0240
+        RTJ      QUIRKU7                ROUNDS AND RENORMALIZES RESULT      3Q0250
+        RTJ      REPACK7                REPACKS RESULT AND                  3Q0260
+        STA      ACC+2                  STORES IN ACC+2                     3Q0270
C1       LDA      ACC                    LOADS A7 REAL                       3Q0280
```

```
          FAD        **         FLOATING ADDS B7 REAL             3Q0290
          STA        ACC        STORES RESULT IN ACC              3Q0300
          SLJ        Q1Q02770                                     3Q0310
          ENTRY      Q1Q03770   ROUTINE FOR RANGE (-) RANGE,      4Q0000
Q1Q03770  SLJ        **         I.E. A7(-)B7. A7 IN RANGE         4Q0010
          LDA        *          ACCUMULATOR.                      4Q0020
          ALS        +24                                          4Q0030
          INA        -1                                           4Q0040
          SAU        *+1                                          4Q0050
+         ENA     7  **                                           4Q0060
          SAL        D1                                           4Q0070
          INA        +1                                           4Q0080
          SAU        D2                                           4Q0090
          INA        +1                                           4Q0100
          SAU        D3                                           4Q0110
          LDA        ACC+2      LOAD A7 UPPER                     4Q0120
          STA        B          STORE IN B                        4Q0130
D2        LAC        **         LOAD COMPLEMENT OF B7 LOWER       4Q0140
+         RTJ        UNPACK7    UNPACK, ADD, ROUND,RENORMALIZE    4Q0150
+         RTJ        RNGAD7     RESULT                            4Q0160
+         RTJ        QUIRKU7                                      4Q0170
+         RTJ        REPACK7                                      4Q0180
+         STA        ACC+2      RESULT TO ACC+2                   4Q0190
          LDA        ACC+1      LOAD A7 LOWER                     4Q0200
          STA        B          STORE IN B                        4Q0210
D3        LAC        **         LOAD B7 UPPER COMPLEMENT          4Q0220
+         RTJ        UNPACK7    PERFORM RANGE ADDITION            4Q0230
+         RTJ        RNGAD7                                       4Q0240
+         RTJ        QUIRKL7                                      4Q0250
+         RTJ        REPACK7                                      4Q0260
+         STA        ACC+1      STORE IN ACC+1                    4Q0270
D1        LDA        ACC        LOAD A7 REAL                      4Q0280
          FSB        **         FLOATING SUBTRACT B7 REAL         4Q0290
          STA        ACC        RESULT TO ACC                     4Q0300
          SLJ        Q1Q03770                                     4Q0310
```

```
         ENTRY      Q1Q04770            ROUTINE FOR RANGE (*) RANGE,        5Q00000
Q1Q04770 SLJ        **                  I.E. A7*B7. A7 IN RANGE ACC.        5Q00010
         LDA        *                                                       5Q0002
         ALS        +24                                                     5Q00030
         INA        -1                                                      5Q00040
         SAU        *+1                                                     5Q00050
+        ENA     7  **                                                      5Q00060
         SAL        RE                                                      5Q00070
         INA        +1                                                      5Q00080
         SAU        E21                                                     5Q00090
         SAU        E22                                                     5Q00100
         INA        +1                                                      5Q00110
         SAU        E31                                                     5Q00120
         SAU        E32                                                     5Q00130
E21      LDA        **                  CALCULATES ALL POSSIBLE PRODUCTS    5Q00140
         STA        EM1                 USING REAL ARITHMETIC               5Q00150
         FMU        ACC+1                                                   5Q00160
         STA        A                   A7 LOWER(*)B7 LOWER. STORE IN A.    5Q00170
E22      LDA        **                  A7 UPPER(*)B7 LOWER.                5Q00180
         FMU        ACC+2               STORE IN A+1                        5Q00190
         STA        A+1                                                     5Q00200
E31      LDA        **                                                      5Q00210
         STA        EM2                                                     5Q00220
         FMU        ACC+1               A7 LOWER(*)B7 UPPER                 5Q00230
         STA        A+2                 STORE IN A+2                        5Q00240
E32      LDA        **                                                      5Q00250
         FMU        ACC+2               A7 UPPER(*)B7 UPPER                 5Q00260
         STA        A+3                 STORE IN A+3                        5Q00270
         LDA        A                   THIS PORTION OF ROUTINE SEARCHES    5Q00280
         FSB        A+1                 FOR THE MAXIMUM OF THE FOUR         5Q00290
         AJP     2  ETST1               PRODUCTS                            5Q00300
         LDA        A+1                                                     5Q00310
         FSB        A+2                                                     5Q00320
         AJP     2  ETST3                                                   5Q00330
         LDA        A+2                                                     5Q00340
```

```
         FSB       A+3                                        5Q0350
         AJP    2  RMAX3                                      5Q0360
         SLJ       RMAX4                                      5Q0370
ETST1    LDA       A                                          5Q0380
         FSB       A+2                                        5Q0390
         AJP    2  ETST2                                      5Q0400
         LDA       A+2                                        5Q0410
         FSB       A+3                                        5Q0420
         AJP    2  RMAX3                                      5Q0430
         SLJ       RMAX4                                      5Q0440
ETST2    LDA       A                                          5Q0450
         FSB       A+3                                        5Q0460
         AJP    2  RMAX1                                      5Q0470
         SLJ       RMAX4                                      5Q0480
ETST3    LDA       A+1                                        5Q0490
         FSB       A+3                                        5Q0500
         AJP    2  RMAX2                                      5Q0510
         SLJ       RMAX4                                      5Q0520
RMAX1    LDA       ACC+1         MAX IN A                     5Q0530
         STA       B                                          5Q054
         LDA       EM1                                        5Q0550
         SLJ       M                                          5Q056
RMAX2    LDA       ACC+2         MAX IN A+1                   5Q0570
         STA       B                                          5Q058
         LDA       EM1                                        5Q0590
         SLJ       M                                          5Q060
RMAX3    LDA       ACC+1         MAX IN A+2                   5Q0610
         STA       B                                          5Q062
         LDA       EM2                                        5Q0630
         SLJ       M                                          5Q064
RMAX4    LDA       ACC+2         MAX IN A+3                   5Q0650
         STA       B                                          5Q066
         LDA       EM2                                        5Q0670
         SLJ       M                                          5Q068
M        RTJ       UNPACK7       RECALCULATE MAX PRODUCT USING  5Q0690
```

```
+         RTJ        RNGMU7           RANGE ARITHMETIC          5Q0700
+         RTJ        QUIRKU7                                    5Q0710
+         RTJ        REPACK7                                    5Q0720
+         STA        MTEMPU           STORE IN MTEMPU           5Q0740
          LDA        A                FIND MIN PRODUCT          5Q0750
          FSB        A+1                                        5Q0760
          AJP    3   EMST1                                      5Q0770
          LDA        A+1                                        5Q0780
          FSB        A+2                                        5Q0790
          AJP    3   EMST3                                      5Q0800
          LDA        A+2                                        5Q0810
          FSB        A+3                                        5Q0820
          AJP    3   RMIN3                                      5Q0830
          SLJ        RMIN4                                      5Q0840
EMST1     LDA        A                                          5Q0850
          FSB        A+2                                        5Q0860
          AJP    3   EMST2                                      5Q0870
          LDA        A+2                                        5Q0880
          FSB        A+3                                        5Q0890
          AJP    3   RMIN3                                      5Q0900
          SLJ        RMIN4                                      5Q0910
EMST2     LDA        A                                          5Q0920
          FSB        A+3                                        5Q0930
          AJP    3   RMIN1                                      5Q0940
          SLJ        RMIN4                                      5Q0950
EMST3     LDA        A+1                                        5Q0960
          FSB        A+3                                        5Q0970
          AJP    3   RMIN2                                      5Q0980
          SLJ        RMIN4                                      5Q0990
RMIN1     LDA        ACC+1            MIN IN A.                 5Q1000
          STA        B                                          5Q101
          LDA        EM1                                        5Q1020
          SLJ        NM                                         5Q1030
RMIN2     LDA        ACC+2            MIN IN A+1                5Q1040
```

```
         STA      B                                                5Q105
         LDA      EM1                                              5Q1060
         SLJ      NM                                               5Q1070
RMIN3    LDA      ACC+1           MIN IN A+2                       5Q1080
         STA      B                                                5Q109
         LDA      EM2                                              5Q1100
         SLJ      NM                                               5Q1110
RMIN4    LDA      ACC+2           MIN IN A+3                       5Q1120
         STA      B                                                5Q113
         LDA      EM2                                              5Q1140
NM       RTJ      UNPACK7         RECALCULATE MIN PRODUCT USING    5Q1150
+        RTJ      RNGMU7          RANGE ARITHMETIC                 5Q1160
+        RTJ      QUIRKL7                                          5Q1170
+        RTJ      REPACK7                                          5Q1180
+        STA      ACC+1           STORE IN ACC+1                   5Q1200
         LDA      MTEMPU          LOAD MTEMPU                      5Q1201
         STA      ACC+2           STORE IN ACC+2                   5Q1202
RE       LDA      ACC                                              5Q1210
         FMU      **              CALCULATE REAL PRODUCT           5Q1220
         STA      ACC             STORE IN ACC                     5Q1230
         SLJ      Q1Q04770                                         5Q1240
         ENTRY    Q1Q05770        ROUTINE FOR RANGE(/)RANGE,       6Q0000
Q1Q05770 SLJ      **              I.E. A7/B7. A7 IN RANGE ACC      6Q0010
         LDA      *                                                6Q002
         ALS      +24                                              6Q0030
         INA      -1                                               6Q0040
         SAU      *+1                                              6Q0050
+        ENA    7 **                                               6Q0060
         SAL      FPACKUP                                          6Q0070
         SAL      FERROR+1                                         6Q0080
         INA      +1                                               6Q0090
         SAU      FTESTLO                                          6Q0100
         SAU      FOKAY                                            6Q0110
         SAL      F21                                              6Q0120
         SAL      F22                                              6Q0130
```

```
         INA      +1                                                   6Q0140
         SAU      FTESTHI                                              6Q0150
         SAU      FSTORE                                               6Q0160
         SAL      F31                                                  6Q0170
         SAL      F32                                                  6Q0180
FTESTLO  LDA      **              TESTS TO SEE IF DIVISOR              6Q0190
         AJP    2 FOKAY           BRACKETS ZERO, IF SO ERROR777        6Q0200
FTESTHI  LDA      **              IS CALLED AND DIAGNOSTIC IS          6Q0210
         AJP    3 FOKAY           PRINTED. IF NOT, ROUTINE IS          6Q0220
FERROR   RTJ      ERROR777        EXECUTED                             6Q0230
+        0        Q1005770                                             6Q0240
         0        **                                                   6Q0250
         BCD      1DIVISION                                            6Q0260
FOKAY    LDA      **              CALCULATE ALL POSSIBLE QUOTIENTS     6Q0270
         STA      FM1                                                  6Q0280
F21      LDA      ACC+1                                                6Q0290
         FDV      **                                                   6Q0300
         STA      A               A7 LOWER(/)B7 LOWER IN A             6Q0310
F22      LDA      ACC+2                                                6Q0320
         FDV      **                                                   6Q0330
         STA      A+1             A7 UPPER(/)B7 LOWER IN A+1           6Q0340
FSTORE   LDA      **                                                   6Q0350
         STA      FM2                                                  6Q0360
F31      LDA      ACC+1                                                6Q0370
         FDV      **                                                   6Q0380
         STA      A+2             A7 LOWER(/)B7 UPPER IN A+2           6Q0390
F32      LDA      ACC+2                                                6Q0400
         FDV      **                                                   6Q0410
         STA      A+3             A7 UPPER(/)B7 UPPER IN A+3           6Q0420
         LDA      A               SEARCH FOR MAXIMUM QUOTIENT          6Q0430
         FSB      A+1                                                  6Q0440
         AJP    2 FTST1                                                6Q0450
         LDA      A+1                                                  6Q0460
         FSB      A+2                                                  6Q0470
         AJP    2 FTST3                                                6Q0480
```

```
        LDA       A+2                                    6Q0490
        FSB       A+3                                    6Q0500
        AJP    2  FMAX3                                  6Q0510
        SLJ       FMAX4                                  6Q0520
FTST1   LDA       A                                      6Q0530
        FSB       A+2                                    6Q0540
        AJP    2  FTST2                                  6Q0550
        LDA       A+2                                    6Q0560
        FSB       A+3                                    6Q0570
        AJP    2  FMAX3                                  6Q0580
        SLJ       FMAX4                                  6Q0590
FTST2   LDA       A                                      6Q0600
        FSB       A+3                                    6Q0610
        AJP    2  FMAX1                                  6Q0620
        SLJ       FMAX4                                  6Q0630
FTST3   LDA       A+1                                    6Q0640
        FSB       A+3                                    6Q0650
        AJP    2  FMAX2                                  6Q0660
        SLJ       FMAX4                                  6Q0670
FMAX1   LDA       FM1         MAX IN A                   6Q0680
        STA       B                                      6Q069
        LDA       ACC+1                                  6Q0700
        SLJ       MAXR                                   6Q0710
FMAX2   LDA       FM1         MAX IN A+1                 6Q0720
        STA       B                                      6Q073
        LDA       ACC+2                                  6Q0740
        SLJ       MAXR                                   6Q0750
FMAX3   LDA       FM2         MAX IN A+2                 6Q0760
        STA       B                                      6Q077
        LDA       ACC+1                                  6Q0780
        SLJ       MAXR                                   6Q0790
FMAX4   LDA       FM2         MAX IN A+3                 6Q0800
        STA       B                                      6Q081
        LDA       ACC+2                                  6Q0820
MAXR    RTJ       UNPACK7     RECOMPUTE MAX QUOTIENT USING  6Q0840
```

```
+         RTJ        RNGDI7                UN-NORMALIZED RANGE ARITHMETIC    6Q0850
+         RTJ        QUIRKU7                                                 6Q0860
+         RTJ        REPACK7                                                 6Q0870
+         STA        DTEMPU                STORE QUOTIENT UPPER IN DTEMPU    6Q0890
          LDA        A                     SEARCH FOR MIN QUOTIENT           6Q0900
          FSB        A+1                                                     6Q0910
          AJP    3   MFTST1                                                  6Q0920
          LDA        A+1                                                     6Q0930
          FSB        A+2                                                     6Q0940
          AJP    3   MFTST3                                                  6Q0950
          LDA        A+2                                                     6Q0960
          FSB        A+3                                                     6Q0970
          AJP    3   FMIN3                                                   6Q0980
          SLJ        FMIN4                                                   6Q0990
MFTST1    LDA        A                                                       6Q1000
          FSB        A+2                                                     6Q1010
          AJP    3   MFTST2                                                  6Q1020
          LDA        A+2                                                     6Q1030
          FSB        A+3                                                     6Q1040
          AJP    3   FMIN3                                                   6Q1050
          SLJ        FMIN4                                                   6Q1060
MFTST2    LDA        A                                                       6Q1070
          FSB        A+3                                                     6Q1080
          AJP    3   FMIN1                                                   6Q1090
          SLJ        FMIN4                                                   6Q1100
MFTST3    LDA        A+1                                                     6Q1110
          FSB        A+3                                                     6Q1120
          AJP    3   FMIN2                                                   6Q1130
          SLJ        FMIN4                                                   6Q1140
FMIN1     LDA        FM1                   MIN IN A                          6Q1150
          STA        B                                                       6Q116
          LDA        ACC+1                                                   6Q1170
          SLJ        MINR                                                    6Q1180
FMIN2     LDA        FM1                   MIN IN A+1                        6Q1190
```

```
         STA     B                                              6Q120
         LDA     ACC+2                                          6Q1210
         SLJ     MINR                                           6Q1220
FMIN3    LDA     FM2            MIN IN A+2                      6Q1230
         STA     B                                              6Q124
         LDA     ACC+1                                          6Q1250
         SLJ     MINR                                           6Q1260
FMIN4    LDA     FM2            MIN IN A+3                      6Q1270
         STA     B                                              6Q128
         LDA     ACC+2                                          6Q1290
MINR     RTJ     UNPACK7        RECOMPUTE MIN QUOTIENT USING    6Q1310
+        RTJ     RNGDI7         UN-NORMALIZED RANGE ARITHMETIC  6Q1320
+        RTJ     QUIRKL7                                        6Q1330
+        RTJ     REPACK7                                        6Q1340
+        STA     ACC+1          STORE QUOTIENT LOWER IN ACC+1   6Q1360
         LDA     DTEMPU                                         6Q1361
         STA     ACC+2          STORE QUOTIENT UPPER IN ACC+2   6Q1362
FPACKUP  LDA     ACC            CALCULATE REAL QUOTIENT         6Q1370
         FDV     **                                             6Q1380
         STA     ACC            STORE IN ACC                    6Q1390
         SLJ     Q1Q05770                                       6Q1400
         ENTRY   ERROR777                                       1E0000
...ADD.. OCT     0                                              1E0010
FORMAT.  BSS     15                                             1E0020
ERROR777 SLJ     **                                             1E0030
         LDA     *                                              1E004
         ALS     +24                                            1E0050
         SAU     H                                              1E006
         INA     +1                                             1E0070
         SAL     H2                                             1E0080
         INA     +1                                             1E0090
         SAU     H3                                             1E0100
H0       LDA     **                                             1E0110
         SAL     H2+1                                           1E0120
         ALS     +24                                            1E0130
```

```
         SAU      *+1                                               1E0140
+        LDA      **                                                1E0150
         ALS      +24                                               1E0160
         INA      -1                                                1E0170
H1       SAL      ...ADD..                                          1E0180
         ENA      +51                                               1E0190
         ENQ      ..2000                                            1E0200
+        CALL     Q8QINGOT                                          1E0210
         0        0                                                 1E022
         CALL     Q8QGOTTY                                          1E0230
         0        GG                                                1E0240
H2       0        0                                                 1E0250
         1        **                                                1E0260
+        60       +3                                                1E0270
         1        **                                                1E0280
         0        0                                                 1E029
         1        ...ADD..                                          1E0300
GG       CALL     Q8QENGOT                                          1E0310
         ORGR     FORMAT.                                           1E0320
..2000   BCD      5(20H0ERROR IN RANGE7(3) ,A8,28H ROUTINE.         1E0330
         BCD      5 EXECUTION DELETED.//12H ARGUMENT = 3E25         1E0340
         BCD      5 .10//22H CALLED FROM LOCATION 05)               1E0350
         ORGR     *                                                 1E036
+        CALL     ERROR*                                            1E0370
H3       SLJ      **                                                1E0380
         ENTRY    Q1Q10770          ROUTINE TO STORE RANGE ACC.     7Q0000
Q1Q10770 SLJ      **                                                7Q0010
         LDA      *                                                 7Q002
         ALS      +24                                               7Q0030
         INA      -1                                                7Q0040
         SAU      *+1                                               7Q0050
+        ENA    7 **                                                7Q0060
         SAL      G1                                                7Q0070
         INA      +1                                                7Q0080
         SAL      G2                                                7Q0090
```

```
         INA      +1                                                 7Q0100
         SAL      G3                                                 7Q0110
G2       LDA      ACC+1          STORE RANGE LOWER (ACC+1)           7Q0120
         STA      **                                                 7Q0130
G3       LDA      ACC+2          STORE RANGE UPPER (ACC+2)           7Q0140
         STA      **                                                 7Q0150
G1       LDA      ACC            STORE REAL PART  (ACC)              7Q0160
         STA      **                                                 7Q0170
         SLJ      Q1Q10770                                           7Q0180
         ENTRY    Q1Q00710       ROUTINE TO LOAD REAL INTO           8Q0000
Q1Q00710 SLJ      **             RANGE ACCUMULATOR                   8Q0010
         LDA      *                                                  8Q002
         ALS      +24                                                8Q0030
         INA      -1                                                 8Q0040
         SAU      *+1                                                8Q0050
+        ENA    7 **                                                 8Q0060
         SAU      AA1                                                8Q0070
         SAU      AA2                                                8Q0080
         SAU      AA3                                                8Q0090
AA2      LDA      **             LOAD REAL                           8Q0100
         STA      ACC+1          STORE IN ACC+1                      8Q0110
         ENA      0                                                  8Q012
         STA      B                                                  8Q013
AA3      LDA      **             LOAD REAL                           8Q0140
         RTJ      UNPACK7        UN-NORMALIZE ADD ZERO,USING         8Q0150
+        RTJ      RNGAD7         QUIRKU7 ROUTINE FOR RANGE UPPER     8Q0160
+        RTJ      QUIRKU7                                            8Q0170
+        RTJ      REPACK7                                            8Q0180
+        STA      ACC+2          STORE IN ACC+2                      8Q0190
AA1      LDA      **             LOAD REAL                           8Q0200
         STA      ACC            STORE IN ACC.                       8Q0210
         SLJ      Q1Q00710                                           8Q0220
         ENTRY    Q1Q01710                                           9Q0000
Q1Q01710 SLJ      **             ROUTINE TO LOAD REAL COMPLEMENT     9Q0010
         LDA      *              INTO RANGE ACCUMULATOR              9Q0020
```

```
         ALS      +24                                              9Q0030
         INA      -1                                               9Q0035
         SAU      *+1                                              9Q0040
+        ENA    7 **                                               9Q0050
         SAU      *+1                                              9Q0060
         LAC      **              LOAD REAL COMPLEMENT             9Q0070
         STA      C               STORE TEMPORARILY IN C           9Q0080
         RTJ      Q1Q00710        CALL ROUTINE TO LOAD C INTO      9Q0090
         0        C               RANGE ACCUMULATOR                9Q0100
         SLJ      Q1Q01710                                         9Q0110
         ENTRY    Q1Q02710                                        10Q0000
Q1Q02710 SLJ      **              ROUTINE FOR RANGE(+)REAL        10Q0010
         LDA      *               RANGE NUMBER IS IN RANGE        10Q0020
         ALS      +24             ACCUMULATOR                     10Q0030
         INA      -1                                              10Q0040
         SAU      *+1                                             10Q0050
+        ENA    7 **                                              10Q0060
         SAL      *+2                                             10Q0070
+        RTJ      Q1Q10770        STORE RANGE ACCUMULATOR IN AS   10Q0080
         0        AS                                              10Q0090
+        RTJ      Q1Q00710        CONVERT REAL TO RANGE AND       10Q0100
         0        **              PUT IN RANGE ACCUMULATOR        10Q0110
         RTJ      Q1Q02770        CALL ROUTINE FOR RANGE(+)RANGE  10Q0120
         0        AS              ADD. RESULT IS IN RANGE         10Q0130
         SLJ      Q1Q02710        ACCUMULATOR                     10Q0140
         ENTRY    Q1Q03710                                        11Q0000
Q1Q03710 SLJ      **              ROUTINE FOR RANGE(-)REAL        11Q0010
         LDA      *               RANGE NUMBER IS IN RANGE        11Q0020
         ALS      +24             ACCUMULATOR                     11Q0030
         INA      -1                                              11Q0040
         SAU      *+1                                             11Q0050
+        ENA    7 **                                              11Q0060
         SAL      *+2                                             11Q0070
+        RTJ      Q1Q10770        STORE RANGE NUMBER IN   A.      11Q0080
         0        A.                                              11Q0090
```

```
+         RTJ        Q1Q01710             LOAD REAL COMPLEMENT INTO          11Q0100
          0          **                   RANGE ACCUMULATOR                  11Q0110
          RTJ        Q1Q02770             ADD THE TWO RANGE NUMBERS          11Q0120
          0          A.                   RESULT IS IN RANGE ACCUMULATOR     11Q0130
          SLJ        Q1Q03710                                                11Q0140
          ENTRY      Q1Q04710                                                12Q0000
Q1Q04710  SLJ        **                   ROUTINE FOR RANGE(*)REAL           12Q0010
          LDA        *                                                       12Q002
          ALS        +24                                                     12Q0030
          INA        -1                                                      12Q0040
          SAU        *+1                                                     12Q0050
+         ENA      7 **                                                      12Q0060
          SAL        *+2                                                     12Q0070
+         RTJ        Q1Q10770             STORE RANGE ACCUMULATOR INTO       12Q0080
          0          AS.                  AS.                                12Q0090
+         RTJ        Q1Q00710             LOAD REAL INTO RANGE ACCUMULATOR   12Q0100
          0          **                                                      12Q0110
          RTJ        Q1Q04770             MULTIPLY RANGE NUMBERS             12Q0120
          0          AS.                  REAULT IN RANGE ACCUMULATOR        12Q0130
          SLJ        Q1Q04710                                                12Q0140
          ENTRY      Q1Q05710                                                13Q0000
Q1Q05710  SLJ        **                   ROUTINE FOR RANGE(/)REAL           13Q0010
          LDA        *                                                       13Q002
          ALS        +24                                                     13Q0030
          INA        -1                                                      13Q0040
          SAU        *+1                                                     13Q0050
+         ENA      7 **                                                      13Q0060
          SAL        *+2                                                     13Q0070
+         RTJ        Q1Q10770             STORE RANGE ACCUMULATOR INTO       13Q0080
          0          Q.                   Q.                                 13Q0090
+         RTJ        Q1Q00710             LOAD REAL INTO RANGE               13Q0100
          0          **                   ACCUMULATOR                        13Q0110
          RTJ        Q1Q10770             STORE RANGE ACCUMULATOR INTO       13Q0120
          0          SA                   SA                                 13Q0130
          RTJ        Q1Q00770             LOAD RANGE ACCUMULATOR FROM        13Q0140
          0          Q.                   Q.                                 13Q0150
```

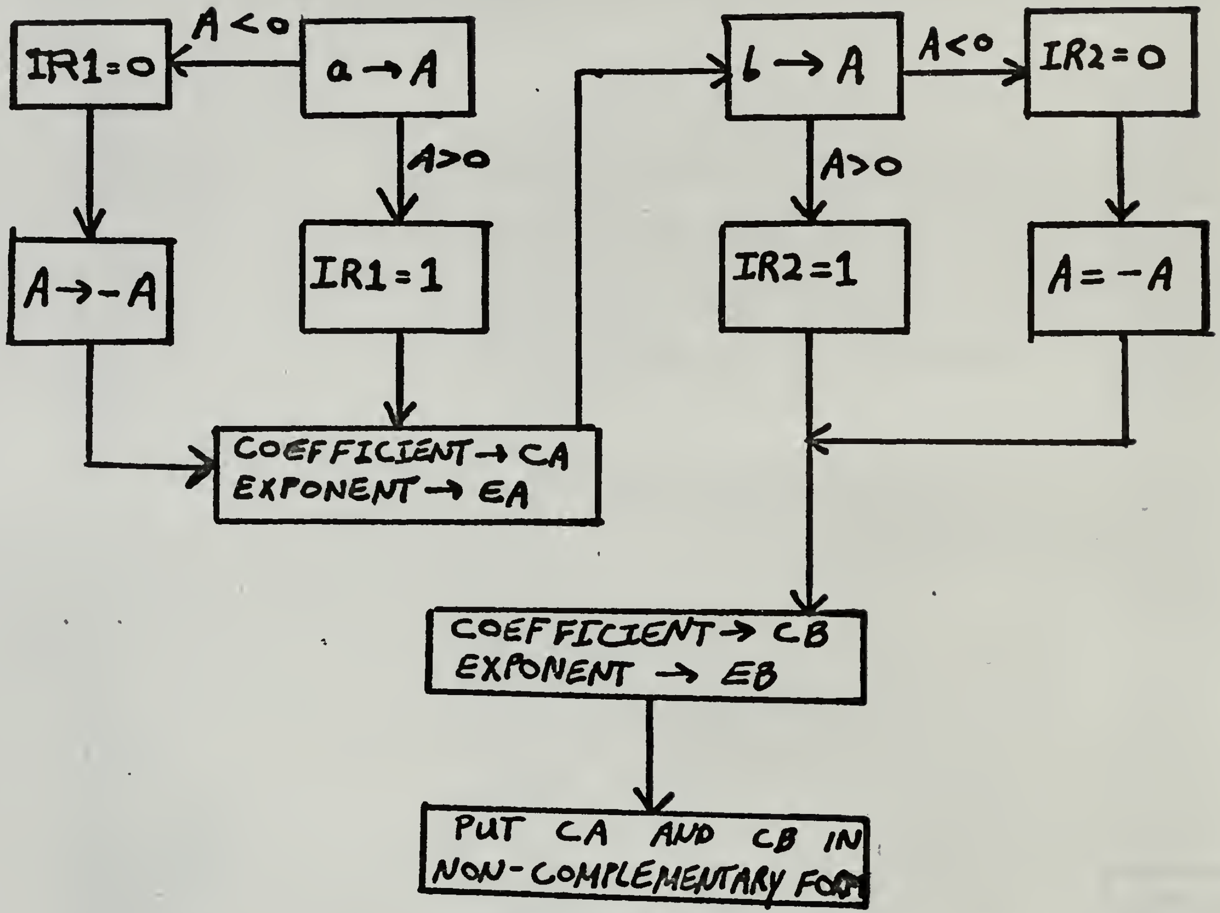
```
         RTJ        Q1Q05770                 RANGE DIVIDE BY RANGE NUMBER      13Q00160
         0          SA                       IN SA                             13Q00170
         SLJ        Q1Q05710                 RESULT IN RANGE ACCUMULATOR       13Q00180
         ENTRY      Q1Q10710                                                   14Q00000
Q1Q10710 SLJ        **                       STORE RANGE INTO REAL             14Q00010
         LDA        *                                                          14Q002
         ALS        +24                                                        14Q00030
         INA        -1                                                         14Q00040
         SAU        *+1                                                        14Q00050
+        ENA     7  **                                                         14Q00060
         SAL        *+1                                                        14Q00070
+        LDA        ACC                      LOAD ACC(REAL PART OF RANGE NO.)  14Q00080
         STA        **                       STORE ACC                         14Q00090
         SLJ        Q1Q10710                                                   14Q00100
         ENTRY      Q1Q10170                                                   15Q00000
Q1Q10170 SLJ        **                       STORE REAL INTO RANGE             15Q00010
         STA        ACC                      STORE REAL INTO ACC               15Q00020
         STA        ACC+1                    STORE REAL INTO ACC+1             1580030
         ENA        +                        PUT ZERO IN A-REGISTER            1580040
         STA        B                        STORE IN B                        1580050
+        LDA        ACC                      LOAD ACC                          1580060
         RTJ        UNPACK7                  UN-NORMALIZED ADD ZERO TO         1580070
+        RTJ        RNGAD7                   ACC, CONSIDERING IT TO BE         1580080
+        RTJ        QUIRKU7                  ADDITION OF UPPER RANGE           15Q00090
+        RTJ        REPACK7                  NUMBERS                           15Q00100
+        STA        ACC+2                    STORE IN ACC+2                    1580110
+        LDA        Q1Q10170                                                   15Q00120
         ALS        +24                                                        1580130
         INA        -1                                                         15Q00140
         SAU        *+1                                                        15Q00150
+        ENA     7  **                                                         15Q00160
         SAL        *+1                                                        15Q00170
         RTJ        Q1Q10770                 STORE RANGE ACCUMULATOR           15Q00180
         0          **                                                         15Q00190
         SLJ        Q1Q10170                                                   15Q00200
         ENTRY      Q1Q00700                                                   16Q00000
```

```
Q1Q00700 SLJ        **         LOAD INTEGER INTO RANGE                16Q0010
         LDA        *                                                 16Q002
         ALS        +24                                               16Q0030
         INA        -1                                                16Q0040
         SAU        *+1                                               16Q0050
+        LDA     7  **         LOAD INTEGER                           16Q0060
         CALL       FLOATF     CONVERT TO FLOATING POINT(REAL)        16Q0070
+        STA        AI         STORE IN AI                            16Q0080
+        RTJ        Q1Q00710   CALL ROUTINE TO LOAD  A(REAL)          16Q0090
         0          AI         INTO RANGE                             16Q0100
         SLJ        Q1Q00700                                          16Q0110
         ENTRY      Q1Q01700                                          17Q0000
Q1Q01700 SLJ        **         LOAD INTEGER COMPLEMENT INTO           17Q0010
         LDA        *          RANGE                                  17Q0020
         ALS        +24                                               17Q0030
         INA        -1                                                17Q0040
         SAU        *+1                                               17Q0050
+        LAC     7  **         LOAD INTEGER COMPLEMENT                17Q0060
         CALL       FLOATF     CONVERT TO REAL                        17Q0070
+        STA        IA         STORE IN IA                            17Q0080
+        RTJ        Q1Q00710   LOAD REAL TO RANGE                     17Q0090
         0          IA                                                17Q0100
         SLJ        Q1Q01700                                          17Q0110
         ENTRY      Q1Q02700                                          18Q0000
Q1Q02700 SLJ        **         ROUTINE FOR RANGE(+)INTEGER            18Q0010
         LDA        *                                                 18Q002
         ALS        +24                                               18Q0030
         INA        -1                                                18Q0040
         SAU        *+1                                               18Q0050
+        ENA     7  **                                                18Q0060
         SAL        *+2                                               18Q0070
+        RTJ        Q1Q10770   STORE RANGE ACCUMULATOR                18Q0080
         0          AUX        IN AUX                                 18Q0090
+        RTJ        Q1Q00700   LOAD INTEGER TO RANGE                  18Q0100
         0          **         ADD THE RANGE NUMBERS                  18Q0110
         RTJ        Q1Q02770                                          18Q0120
         0          AUX                                               18Q0130
```

```
         SLJ      Q1Q02700                                         18Q0140
         ENTRY    Q1Q03700                                         19Q0000
Q1Q03700 SLJ      **                ROUTINE FOR RANGE(-)INTEGER    19Q0010
         LDA      *                                                19Q002
         ALS      +24                                              19Q0030
         INA      -1                                               19Q0040
         SAU      *+1                                              19Q0050
+        ENA    7 **                                               19Q0060
         SAL      *+2                                              19Q0070
+        RTJ      Q1Q10770          STORE RANGE ACCUMULATOR        19Q0080
         0        AUX.              IN AUX.                        19Q0090
+        RTJ      Q1Q01700          LOAD INTEGER COMPLEMENT TO     19Q0100
         0        **                RANGE                          19Q0110
         RTJ      Q1Q02770          ADD RANGE NUMBERS              19Q0120
         0        AUX.                                             19Q0130
         SLJ      Q1Q03700                                         19Q0140
         ENTRY    Q1Q04700                                         20Q0000
Q1Q04700 SLJ      **                ROUTINE FOR RANGE(*)INTEGER    20Q0010
         LDA      *                                                20Q002
         ALS      +24                                              20Q0030
         INA      -1                                               20Q0040
         SAU      *+1                                              20Q0050
+        ENA    7 **                                               20Q0060
         SAL      *+2                                              20Q0070
+        RTJ      Q1Q10770          STORE RANGE ACCUMULATOR        20Q0080
         0        SAUX              IN SAUX                        20Q0090
+        RTJ      Q1Q00700          LOAD INTEGER TO RANGE          20Q0100
         0        **                                               20Q0110
         RTJ      Q1Q04770          MULTIPLY RANGE NUMBERS         20Q0120
         0        SAUX                                             20Q0130
         SLJ      Q1Q04700                                         20Q0140
         ENTRY    Q1Q05700                                         21Q0000
Q1Q05700 SLJ      **                ROUTINE FOR RANGE(/)INTEGER    21Q0010
         LDA      *                                                21Q002
         ALS      +24                                              21Q0030
```

```
         INA       -1                                         21Q00040
         SAU       *+1                                        21Q00050
+        ENA    7  **                                         21Q00060
         SAL       *+2                                        21Q00070
+        RTJ       Q1Q10770        STORE RANGE ACCUMULATOR    21Q00080
         0         QAUX            IN QAUX                    21Q00090
+        RTJ       Q1Q00700        LOAD INTEGER TO RANGE      21Q00100
         0         **                                         21Q00110
         RTJ       Q1Q10770        STORE RANGE ACCUMULATOR    21Q00120
         0         SAUX.           IN SAUX.                   21Q00130
         RTJ       Q1Q00770        LOAD RANGE ACCUMULATOR     21Q00140
         0         QAUX            FROM QAUX                  21Q00150
         RTJ       Q1Q05770        DIVIDE RANGE BY RANGE NUMBER 21Q00160
         0         SAUX.           IN SAUX.                   21Q00170
         SLJ       Q1Q05700                                   21Q00180
         ENTRY     Q1Q10700                                   22Q00000
Q1Q10700 SLJ       **              STORE RANGE INTO INTEGER   22Q00010
         LDA       *                                          22Q002
         ALS       +24                                        22Q00030
         INA       -1                                         22Q00040
         SAU       *+1                                        22Q00050
+        ENA    7  **                                         22Q00055
         SAU       *+2                                        22Q00060
         LDA       ACC             LOAD ACC (REAL PART)       22Q00070
         CALL      XINTF           CONVERT TO INTEGER         22Q00080
+        STA       **              STORE INTEGER              22Q00090
         SLJ       Q1Q10700                                   22Q00100
         ENTRY     Q1Q10070                                   23Q00000
Q1Q10070 SLJ       **              STORE INTEGER INTO RANGE   23Q00010
         CALL      FLOATF          CONVERT INTEGER TO REAL    23Q00020
+        STA       AF              STORE IN AF                23Q00030
         LDA       *-1                                        23Q00040
         ALS       +24                                        23Q00050
         INA       -1                                         23Q00060
         SAU       *+1                                        23Q00070
```

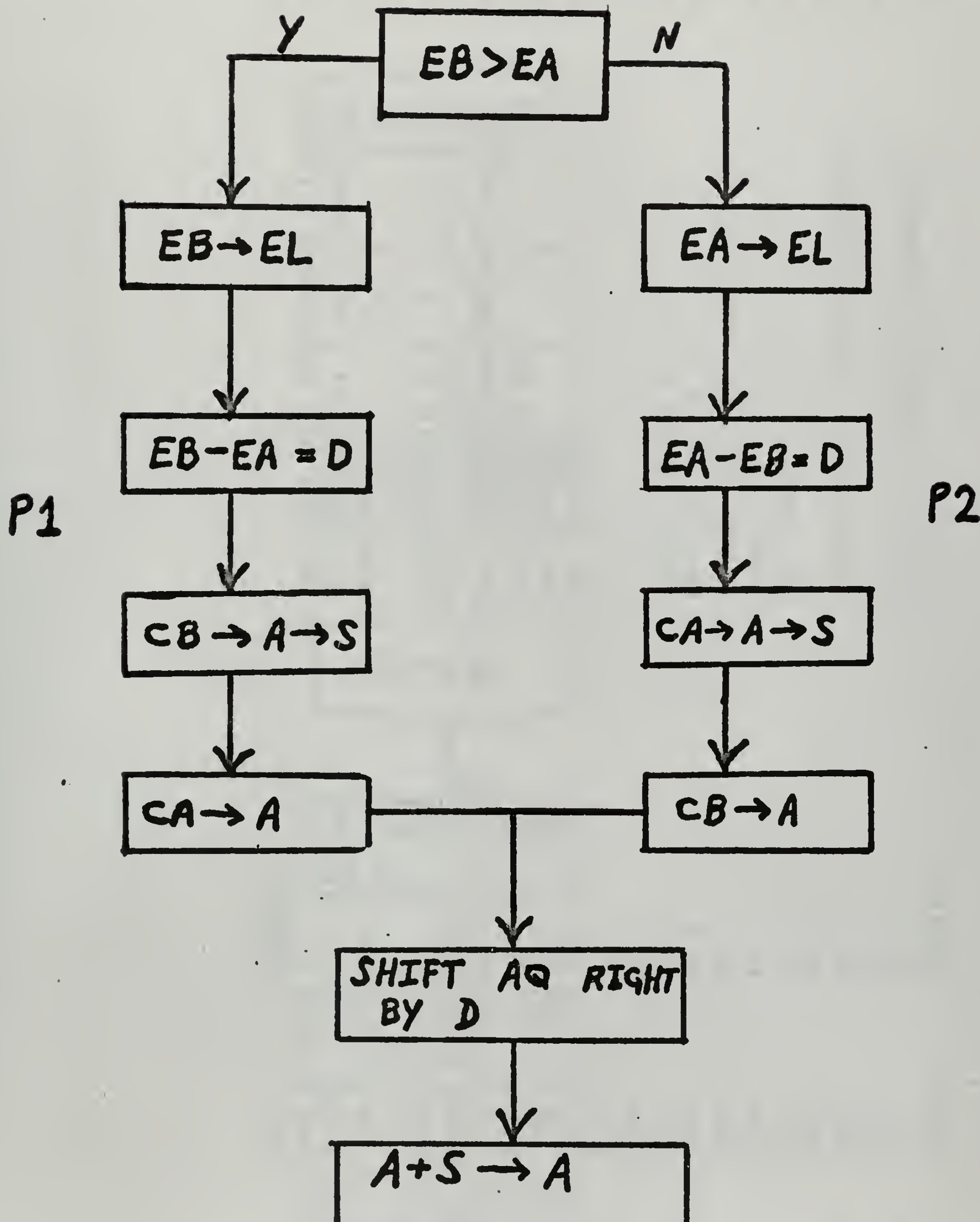
```
+     ENA    7   **                                      23Q00080
      SAL        *+2                                     23Q00090
+     LDA        AF              LOAD AF                 23Q00100
+     RTJ        Q1Q10170        STORE REAL  INTO RANGE  23Q00110
      0          **                                      23Q00120
      SLJ        Q1Q10070                                23Q00130
```



FLOW CHART FOR UNPACK7

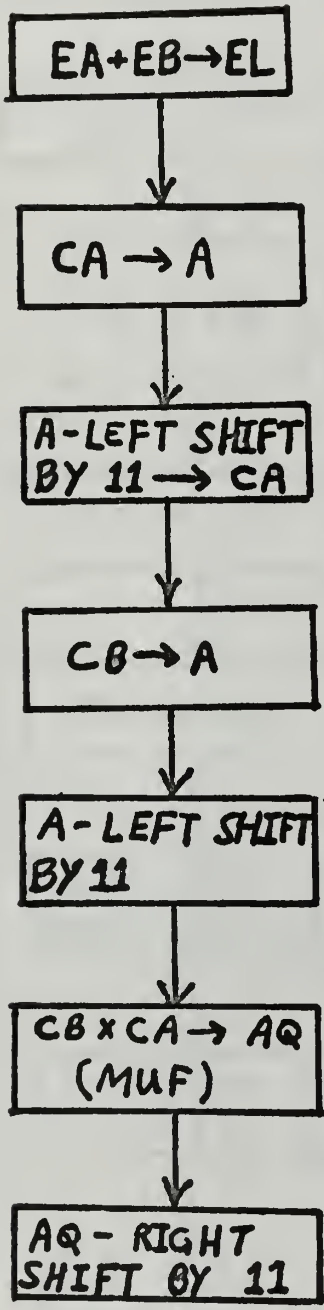
```
         ENTRY     UNPACK7                                              1UN0000
UNPACK7  SLJ       **              UNPACKS TWO FLOATING POINT           1UN0001
         SIU    1  T22             NUMBERS, A AND B                     1UN0002
         SIL    2  T22                                                  1UN0010
         STA       TEMPO           UNPACK A                             1UN0020
         LDQ       TEMPO                                                1UN0030
         QJP    M  T3                                                   1UN0040
         ENI    1  1                                                    1UN005
         SLJ       T4                                                   1UN0060
T3       LQC       TEMPO                                                1UN0070
         ENI    1  0                                                    1UN008
T4       LDL       =0777777777777                                       1UN0090
         STA       CA              COEF OF A IN CA                      1UN0100
         LDL       =0200000000000000                                    1UN0110
         STA       S                                                    1UN012
         LDL       =0377700000000000                                    1UN0130
         ARS       36                                                   1UN0140
         LDQ       S                                                    1UN015
         QJP       T9                                                   1UN0160
         SUB       =02000                                               1UN0170
         STA       EA              EXPONENT OF A IN EA                  1UN0180
         SLJ       T10                                                  1UN0190
T9       SUB       =01777                                               1UN0200
         STA       EA              EXPONENT OF A IN. EA                 1UN0210
T10      LDQ       B               UNPACK B                             1UN0220
         QJP    M  T12                                                  1UN0230
         ENI    2  1                                                    1UN024
         SLJ       T13                                                  1UN0250
T12      LQC       B                                                    1UN0260
         ENI    2  0                                                    1UN027
T13      LDL       =0777777777777                                       1UN0280
         STA       CB              COEF OF B IN CB                      1UN0290
         LDL       =0200000000000000                                    1UN0300
         STA       S                                                    1UN031
         LDL       =0377777777777777                                    1UN0320
         ARS       36                                                   1UN0330
```

```
         LDQ      S                                            1UN034
         QJP      T18                                          1UN0350
         SUB      =O2000                                       1UN0360
         STA      EB              EXPONENT OF B IN EB          1UN0370
         SLJ      T19                                          1UN0380
T18      SUB      =O1777                                       1UN0390
         STA      EB              EXPONENT OF B IN EB          1UN0400
T19      ENA    1 0               PUT UNPACKED NUMBERS IN      1UN0410
         STA      TEMPO           NON-COMPLEMENTARY FORM       1UN0420
+        ENA    2 0                                            1UN043
         ADD      TEMPO                                        1UN0440
         AJP    0 T20                                          1UN0450
         SUB      =O2                                          1UN0460
         AJP    0 T22                                          1UN0470
         ENA    1 0                                            1UN048
         AJP    0 T21                                          1UN0490
         LQC      CB                                           1UN0520
         STQ      CB                                           1UN0530
         SLJ      T22                                          1UN0540
T20      LQC      CB                                           1UN0570
         STQ      CB                                           1UN0580
T21      LQC      CA                                           1UN0610
         STQ      CA                                           1UN0620
T22      ENI    1 **                                           1UN0630
         ENI    2 **                                           1UN0631
T23      SLJ      UNPACK7                                      1UN0632
```



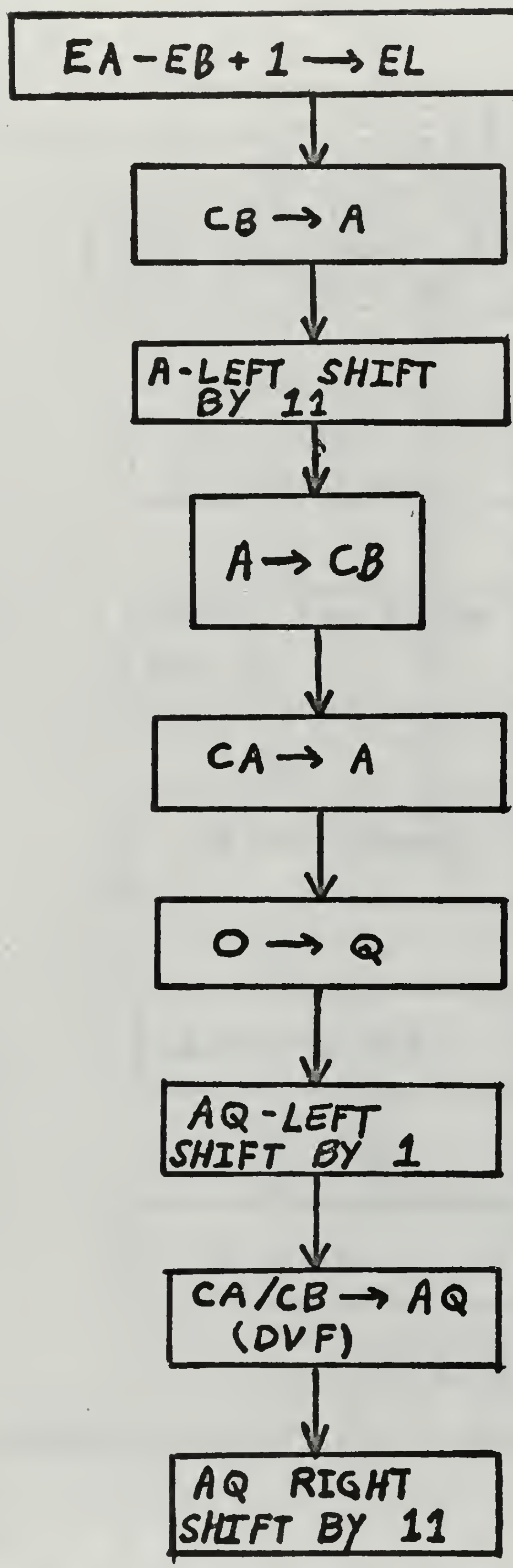
FLOW CHART FOR RNGAD 7

```
         ENTRY     RNGAD7                                              1AD0000
RNGAD7   SLJ       **             ADDS TWO UNPACKED NUMBERS            1AD0010
         LDA       EA                                                  1AD0020
         THS       EB             SEARCH FOR LARGER EXPONENT           1AD0030
         SLJ       P2                                                  1AD0040
P1       LDA       EB             EB LARGER                            1AD0050
         STA       EL             STORE EB IN EL                       1AD0060
         SUB       EA             STORE EXPONENT DIFFERENCE            1AD0070
         SAU       A12            IN A12 UPPER                         1AD0080
         LDA       CB             STORE CB IN S                        1AD0090
         STA       S                                                   1AD010
         LDA       CA                                                  1AD0110
         SLJ       A12            GO TO A12                            1AD0120
P2       LDA       EA             EA LARGER                            1AD0130
         STA       EL             STORE EA IN EL                       1AD0140
         SUB       EB             EXP DIFFERENCE IN A12 UPPER          1AD0150
         SAU       A12                                                 1AD0160
         LDA       CA             STORE CA IN S                        1AD0170
         STA       S                                                   1AD018
         LDA       CB             LOAD CB                              1AD0190
A12      LRS       **             SHIFT A-RIGHT THE DIFFERENCE         1AD0200
         ADD       S              IN EXP, AND ADD COEFFICIENTS         1AD0210
         SLJ       RNGAD7                                              1AD0220
```



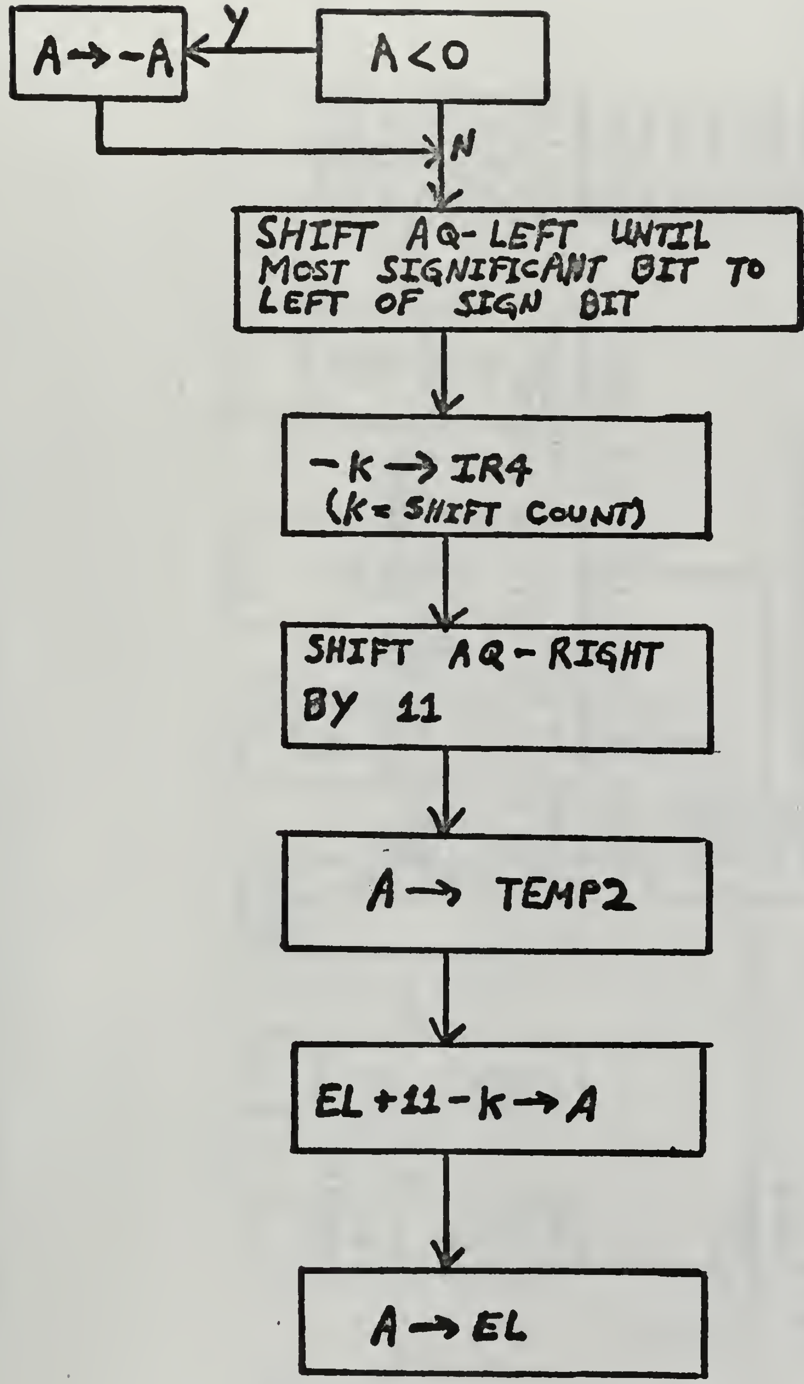
FLOW CHART FOR RNGMU 7

```
         ENTRY     RNGMU7                                    1M0000
RNGMU7   SLJ       **         UN-NORMALIZED MULTIPLY         1M0010
         LDA       EA                                        1M0020
         ADD       EB                                        1M0030
         STA       EL         SUM OF EXPONENTS IN EL         1M0040
         LDA       CA                                        1M0050
         ALS       11         SHIFT CA LEFT 11               1M0060
         STA       CA                                        1M0070
         LDA       CB                                        1M0080
         ALS       11         SHIFT CB LEFT 11               1M0090
         MUF       CA         MULTIPLY COEF FRACTIONALLY     1M0100
         LRS       11         SHIFT A-RIGHT 11               1M0110
         SLJ       RNGMU7                                    1M0120
```



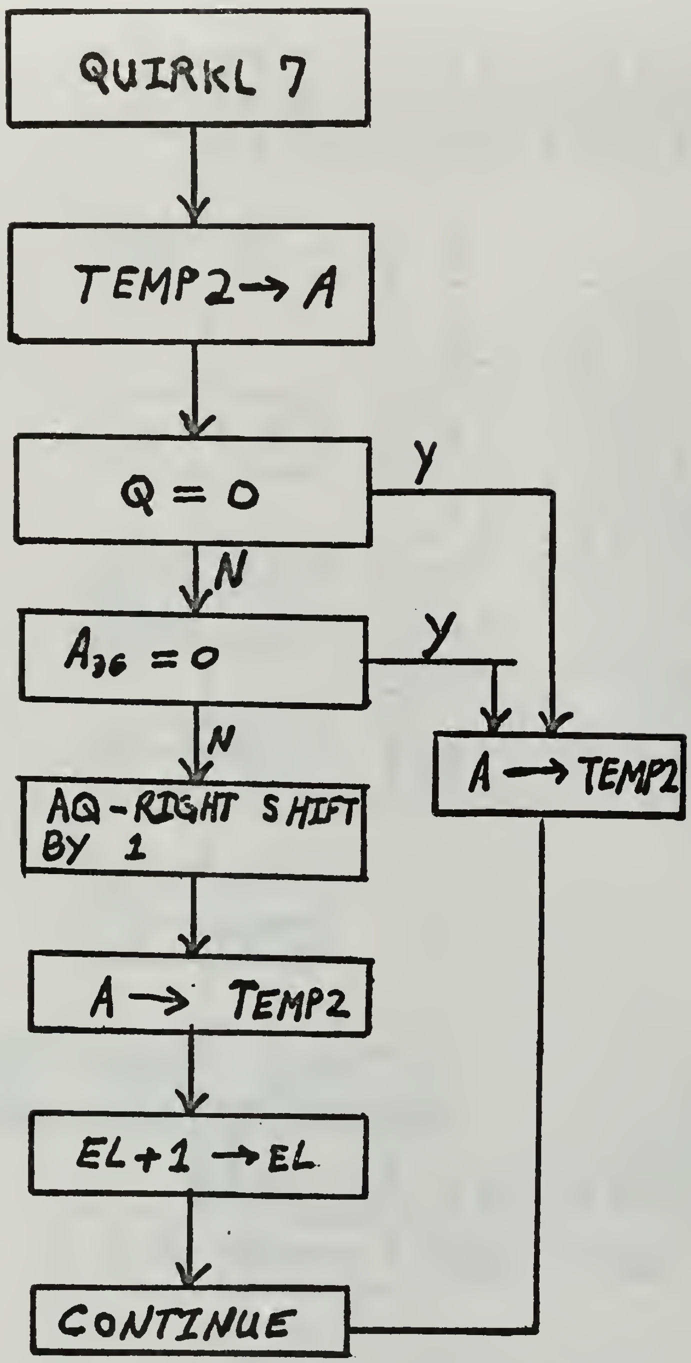
FLOW CHART FOR RNGDI 7

```
         ENTRY    RNGDI7                                        1D0000
RNGDI7   SLJ      **              UN-NORMALIZED DIVISION        1D0010
         LDA      EA                                            1D0020
         SUB      EB                                            1D0030
         INA      +1                                            1D0035
         STA      EL              DIFFERENCE OF EXP +1 IN EL    1D0040
         LDA      CB                                            1D0050
         ALS      11                                            1D0060
         STA      CB                                            1D0070
         LDA      CA              SHIFT CA AND CB EACH RIGHT BY 11  1D0080
         ENQ      0                                             1D009
         LLS      1                                             1D010
         DVF      CB              DIVIDE FRACTIONAL  CA BY CB   1D0110
         LRS      11                                            1D0120
         SLJ      RNGDI7                                        1D0130
```



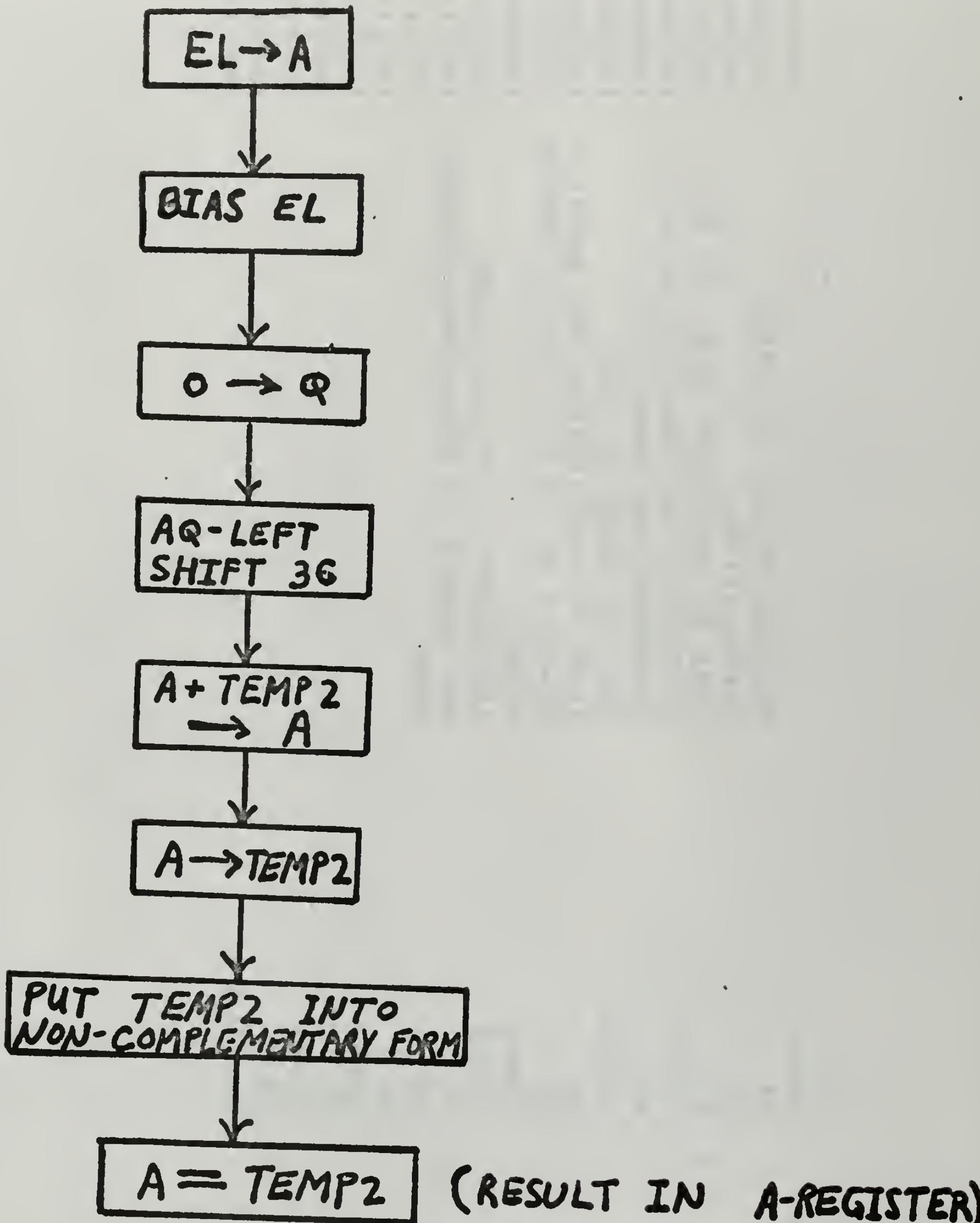
FLOW CHART FOR QUIRKL 7

```
         ENTRY      QUIRKL7                                               1QL0000
QUIRKL7  SLJ        **            ROUTINE TO TRUNCATE AND                 1QL0010
         SIU     4  QL3           NORMALIZE LOWER RANGE NUMBER            1QL0001
         SIU     5  R7                                                    1QL0002
         AJP     2  QL1           IF A-REGISTER IS NEGATIVE,              1QL0020
         ENI     5  0             COMPLEMENT ACCUMULATOR.                 1QL0030
         STA        TEMP4                                                 1QL0040
         LAC        TEMP4                                                 1QL0050
         SLJ        QL2                                                   1QL0060
QL1      ENI     5  1                                                     1QL0070
QL2      ENI     4  47                                                    1QL0080
         SCQ     4  47            SHIFT AQ-REGISTER LEFT UNTIL            1QL0090
         LRS        11            MOST SIGNIFICANT BIT IS TO              1QL0100
         STA        TEMP2         RIGHT OF SIGN BIT. THEN SHIFT           1QL0105
         INI     4  -47           AQ RIGHT BY 11, AND STORE.              1QL0110
         ENA     4  0                                                     1QL012
         INA        11                                                    1QL0130
         ADD        EL            ADJUST EL  DUE TO SHIFT, THAT           1QL0140
         STA        EL            IS, NORMALIZE EXPONENT                  1QL0150
QL3      ENI     4  **                                                    1QL0160
         SLJ        QUIRKL7                                               1QL0170
```



FLOW CHART FOR QUIRKU 7

```
         ENTRY     QUIRKU7                                                        1QU0000
QUIRKU7  SLJ       **                            ROUTINE TO ROUND AND NORMALIZE   1QU0010
         RTJ       QUIRKL7                       UPPER RANGE NUMBER.              1QU0020
+        LDA       TEMP2                         CALL ROUTINE FOR NORMALIZING     1QU0030
         QJP    0  QU1                           LOWER RANGE NUMBERS,AND LOAD     1QU0040
         ADD       =01                           A-REG. WITH COEF. IF Q-REG IS    1QU0050
         LDQ       =0100000000000                ZERO,GO TO QU1. IF NOT,ADD 1     1QU0060
         STL       BIT                           TO COEF.                         1QU0070
         LDQ       BIT                           IF OVERFLOW INTO 37TH BIT,       1QU0080
         QJP    0  QU1                           SHIFT AQ RIGHT BY ONE,AND STORE, 1QU0090
         LRS       1                             THEN INCREASE EL BY ONE.         1QU0100
         STA       TEMP2                         OTHERWISE,STORE A-REQ IN TEMP2.  1QU0110
         LDA       EL                                                             1QU0120
         ADD       1                                                              1QU013
         STA       EL                                                             1QU0140
         SLJ       QU2                                                            1QU0150
QU1      STA       TEMP2                                                          1QU0160
QU2      SLJ       QUIRKU7                                                        1QU0170
```



FLOW CHART FOR REPACK 7

```
         ENTRY       REPACK7                                          1RE0000
REPACK7  SLJ         **            REPACKS RESULT OF RANGE            1RE0010
         LDA         EL            CALCULATION.                       1RE0020
         AJP    2    R1            LOAD EL AND JUMP TO R1 IF POS.     1RE0030
         ADD         =01777        OTHERWISE ADD NEGATIVE BIAS TO     1RE0040
         SLJ         R2            EXPONENT AND JUMP TO R2            1RE0050
R1       ADD         =02000        ADD POS BIAS TO EXP.               1RE0060
R2       ENQ         +0            ZERO IN Q-REGISTER, AND SHIFT      1RE0070
         LLS         36            AQ-LEFT 36 TO PUT EXP IN PROPER    1RE0080
         ADD         TEMP2         PLACE.                             1RE0090
         STA         TEMP2         ADD COEFFICIENT,AND STORE          1RE0100
         ENA    5    0             PUT NUMBER IN NON-COMPLEMENTARY    1RE0110
         AJP    0    R3            FORM                               1RE0120
         SLJ         R4                                               1RE0130
R3       LAC         TEMP2                                            1RE0140
         SLJ         R7                                               1RE0150
R4       LDA         TEMP2                                            1RE0160
R7       ENI    5    **                                               1RE0170
         SLJ         REPACK7                                          1RE0180
         END
```

APPENDIX II

SUBROUTINES FOR USE WITH QRANGE7 [5]

1. ABS7
2. SQRT7
3. INT7
4. Q0Q06700
5. Q2Q07770

```
         IDENT     ABS7                                        ABS70000
         ENTRY     ABS7          ROUTINE FOR ABSOLUTE VALUE    ABS70010
RNG7S777 BLOCK     4               OF A WHERE A IS A RANGE     ABS70060
         COMMON    ACC(3),B        NUMBER                      ABS70040
ABS7     SLJ       **                                          ABS70050
         LDA       ABS7                                        ABS70060
         ALS       +24                                         ABS70070
         SAU       *+2                                         ABS70080
+        INA       +1                                          ABS70090
         SAL       EXIT+2                                      ABS70100
+        LDA       **                                          ABS70110
         ALS       +24                                         ABS70120
+        SAU       *+1                                         ABS70130
         SIL     1 EXIT                                        ABS70140
+        ENI     1 **                                          ABS70150
         LDA     1 +1                                          ABS70160
         AJP     3 NEG           A(2) .LT. ZERO                ABS70170
         STA       ACC+1         A(2) .GT. ZERO                ABS70180
         LDA     1 +2                                          ABS70190
         STA       ACC+2                                       ABS70200
         SLJ       EXIT                                        ABS70210
NEG      SCM       =0777777777777777                           ABS702Z0
         STA       ACC+2         STORE ABSF(A(2)) IN ACC+2     ABS70230
         LDA     1 +2                                          ABS70240
         AJP     2 POS           A(3) .GT. ZERO                ABS70250
         SCM       =0777777777777777  A(3) .LT. ZERO           ABS70260
         STA       ACC+1                                       ABS70270
         SLJ       EXIT                                        ABS70280
POS      THS       ACC+2         A(2) )LT) ZERO .LT. A(3)      ABS70290
         SLJ       *+3                                         ABS70300
+        ENA       0                                           ABS7031
         STA       ACC+1                                       ABS70320
+        SLJ       EXIT                                        ABS70330
+        STA       ACC+2                                       ABS70340
         SLJ       *-2                                         ABS70350
EXIT     LDA     1 +0                                          ABS70360
         ENI     1 **                                          ABS70370
+        AJP     2 *+1                                         ABS70380
         SCM       =0777777777777777                           ABS70390
+        STA       ACC                                         ABS7400
         SLJ       **                                          ABS70410
```

```
         IDENT     SQRT7                                   SQRT0000
         ENTRY     SQRT7                                   SQRT0010
RNG7S777 BLOCK     4                                       SQRT0020
         COMMON    ACC(3),B                                SQRT0030
TACC     BSS       3                                       SQRT0040
SQRT7    SLJ       **                                      SQRT0050
         LDA       *                                       SQRT0060
         ALS       +24                                     SQRT0070
         SAU       *+2                                     SQRT0080
+        INA       1                                       SQRT0090
         SAL       EXIT                                    SQRT0100
         LDA       **                                      SQRT0110
         ALS       +24                                     SQRT0120
         SAL       *+1                                     SQRT0130
         SAL       SQRTERR                                 SQRT0140
+        CALL      Q1Q00770                                SQRT0150
         0         **                                      SQRT0160
         CALL      Q1Q10770                                SQRT0170
         0         TACC                                    SQRT0180
         LDA       TACC+1                                  SQRT0190
         SIU     1 EXIT                                    SQRT0200
         AJP     M SQRTERR                                 SQRT0210
         LDA       TACC                                    SQRT0220
         AJP     Z GO                                      SQRT0225
         LRS       36                                      SQRT0230
+        THS       EXPONCHK                                SQRT0235
         INA       1                                       SQRT0240
         INA       1777B                                   SQRT0250
         LRS       1                                       SQRT0260
         SAU       ADJUSTEX+1                              SQRT0270
         ENA       1   B                                   SQRT0280
         LRS       11                                      SQRT0290
         STQ       ERAS                                    SQRT0300
         LDA       A                                       SQRT0310
         FAD       ERAS                                    SQRT0320
         STA       ERAS1                                   SQRT0330
```

```
         LDA        C                                                   SQRT0340
         FDV        ERAS1                                               SQRT0350
         FAD        A                                                   SQRT0360
         STA        ERAS1                                               SQRT0370
         LDA        ERAS                                                SQRT0380
         FDV        ERAS1                                               SQRT0390
         FAD        ERAS1                                               SQRT0400
         SUB        DIV2                                                SQRT0410
         STA        ERAS1                                               SQRT0420
         LDA        ERAS                                                SQRT0430
         FDV        ERAS1                                               SQRT0440
         FAD        ERAS1                                               SQRT0450
         LRS        36                                                  SQRT0460
         RTJ        ADJUSTEX                                            SQRT0470
GO       STA        TACC                                                SQRT0480
         ENI     1  1                                                   SQRT0490
LOOP     LDA     1  TACC+1                                              SQRT0500
         AJP     Z  GOE                                                 SQRT0510
         LRS        36                                                  SQRT0520
+        THS        EXPONCHK                                            SQRT0530
         INA        1                                                   SQRT0540
         INA        1777B                                               SQRT0550
         LRS        1                                                   SQRT0560
         SAU        ADJUSTEX+1                                          SQRT0570
         ENA        1    B                                              SQRT0580
         LRS        11                                                  SQRT0590
         STQ        ERAS                                                SQRT0600
         LDA        ERAS                                                SQRT0610
         STA        ERAS+1                                              SQRT0620
         ENA        0                                                   SQRT0630
         STA        B                                                   SQRT0640
         LDA        ERAS                                                SQRT0650
         CALL       UNPACK7                                             SQRT0660
+        CALL       RNGAD7                                              SQRT0670
+        CALL       QUIRKU7                                             SQRT0680
+        CALL       REPACK7                                             SQRT0690
```

```
+       STA     ERAS+2                                  SQRT0700
+       CALL    Q1Q00770                                SQRT0710
        0       A                                       SQRT0720
        CALL    Q1Q02770                                SQRT0730
        0       ERAS                                    SQRT0740
        CALL    Q1Q10770                                SQRT0750
        0       ERAS1                                   SQRT0760
        CALL    Q1Q00770                                SQRT0770
        0       C                                       SQRT0780
        CALL    Q1Q05770                                SQRT0790
        0       ERAS1                                   SQRT0880
        CALL    Q1Q02770                                SQRT0810
        0       A                                       SQRT0820
        CALL    Q1Q10770                                SQRT0830
        0       ERAS1                                   SQRT0840
        CALL    Q1Q00770                                SQRT0850
        0       ERAS                                    SQRT0860
        CALL    Q1Q05770                                SQRT0870
        0       ERAS1                                   SQRT0880
        CALL    Q1Q02770                                SQRT0890
        0       ERAS1                                   SQRT0900
        LDA     ACC                                     SQRT0910
        SUB     DIV2                                    SQRT0920
        STA     ERAS1                                   SQRT0930
        LDA     ACC+1                                   SQRT0940
        SUB     DIV2                                    SQRT0950
        STA     ERAS1+1                                 SQRT0960
        LDA     ACC+2                                   SQRT0970
        SUB     DIV2                                    SQRT0980
        STA     ERAS1+2                                 SQRT0990
+       CALL    Q1Q00770                                SQRT1000
        0       ERAS                                    SQRT1005
        CALL    Q1Q05770                                SQRT1010
        0       ERAS1                                   SQRT1020
        CALL    Q1Q02770                                SQRT1030
        0       ERAS1                                   SQRT1040
```

```
         LDA     1  ACC+1                                          SQRT1050
         LRS        36                                             SQRT1060
         RTJ        ADJUSTEX                                       SQRT1070
GOE      STA     1  TACC+1                                         SQRT1080
         IJP     1  LOOP                                           SQRT1090
         CALL       Q1Q00770                                       SQRT1100
         0          TACC                                           SQRT1110
EXIT     ENI     1  **                                             SQRT1120
         SLJ        **                                             SQRT1130
ADJUSTEX SLJ        **                                             SQRT1140
         INA        -4000B                                         SQRT1150
         INA        **                                             SQRT1160
+        AJP     Z  NORM                                           SQRT1170
         INA        1777B                                          SQRT1180
         LLS        36                                             SQRT1190
         SLJ        ADJUSTEX                                       SQRT1200
NORM     INA        1                                              SQRT1210
         SLJ        ADJUSTEX+2                                     SQRT1220
SQRTERR  CALL       ERROR777                                       SQRT1230
+        0          SQRT7                                          SQRT1240
         0          **                                             SQRT1250
         BCD        1 SQRT7                                        SQRT1260
+        SLJ        EXIT                                           SQRT1270
ERAS     BSS        3                                              SQRT1280
ERAS1    BSS        3                                              SQRT1290
DIV2     OCT        1000000000000                                  SQRT1300
EXPONCHK OCT        1777                                           SQRT1310
A        OCT        2002613436725763                               SQRT1320
         OCT        2  2613436725760                               SQRT1330
         OCT        2  2613436725764                               SQRT1340
C        OCT        5773356313766036                               SQRT1350
         OCT        5773356313766033                               SQRT1360
         OCT        5773356313766037                               SQRT1370
         END                                                       SQRT1380
```

```
         IDENT     INT7                                            INT70010
         ENTRY     INT7                                            INT70020
RNG7S777 BLOCK     4                                               INT70030
         COMMON    ACC(3),B                                        INT70040
INT7     SLJ       **                TRUNCATE RANGE7 NUMBER AND    INT70050
         SIU     1 EXIT                RETURN AS A RANGE7 NUMBER   INT70060
         LIU     1 *-1                                             INT70070
         SIU     1 *+2                                             INT70080
         INI     1 1                                               INT70090
         SIL     1 EXIT                                            INT70100
         LIU     1 **                                              INT70110
         LDA     1 0                                               INT70120
         CALL      INTF                                            INT70130
+        STA       ACC                                             INT70140
         INI     1 1                                               INT70150
         LDA     1 0                                               INT70160
         CALL      INTF                                            INT70170
+        STA       ACC+1                                           INT70180
         INI     1 1                                               INT70190
         LDA     1 0                                               INT70200
         CALL      INTF                                            INT70210
+        STA       ACC+2                                           INT70220
         LDA       ACC                                             INT70230
EXIT     ENI     1 **                                              INT70240
         SLJ       **                                              INT70250
         END                                                       INT70260
```

```
         IDENT   Q0Q06700
RNG7S777 BLOCK   4
         COMMON  ACC(3),B
Q        BSS     4
         ENTRY   Q0Q06700          COMPLEMENT RANGE ACCUMULATOR    Q3000010
Q0Q06700 SLJ     **                                                Q3000020
         LDA     ACC+2                                             Q3000030
         STA     Q+3                                               Q3000040
         LAC     ACC+1                                             Q3000050
         STA     ACC+2                                             Q3000060
         LAC     Q+3                                               Q3000070
         STA     ACC+1                                             Q3000080
         LAC     ACC                                               Q3000090
         STA     ACC                                               Q3000100
         SLJ     Q0Q06700                                          Q3000110
         END
```

```
          IDENT         Q2Q07770                                         Q2770000
          ENTRY         Q2Q07770      ROUTINE FOR X**I, WHERE            Q2770010
RNG7S777  BLOCK         4             X IS RANGE AND I IS INTEGER        Q2770030
          COMMON        ACC(3),B                                         Q2770040
ONE       DEC           1.                                               Q2770050
          DEC           1.                                               Q2770060
          DEC           1.                                               Q2770070
X         BSS           3                                                Q2770080
XTO2N     BSS           3                                                Q2770090
I         BSS           1                                                Q2770100
ISAVE     BSS           1                                                Q2770110
Q2Q07770  SLJ           **                                               Q2770120
          SIU     1     EXIT                                             Q2770130
          LDA           *-1                                              Q2770140
          ALS           +24                                              Q2770150
          INA           -1                                               Q2770160
          SAU           START                                            Q2770170
          INA           -1                                               Q2770180
          SAU           START+1                                          Q2770190
START     ENA     7     **            ADDRESS OF I                       Q2770200
          SAU           NOW                                              Q2770210
          ENA     7     **            ADDRESS OF X                       Q2770220
          SAL           *+1                                              Q2770230
+         CALL          Q1Q00770                                         Q2770240
          0             **                                               Q2770250
NOW       LDQ           **            PUT I IN Q-REGISTER                Q2770260
          QJP     P     PROC          ERROR EXIT IF I IS NEGATIVE        Q2770270
          LDA           ACC+1         AND ZERO IS IN RANGE OF X.         Q2770280
          AJP     P     PROC                                             Q2770290
          LDA           ACC+2                                            Q2770300
          AJP     M     PROC                                             Q2770310
ERREXT    CALL          ERROR777      NOTE.  0**0 IS EQUAL TO ONE        Q2770320
+         0             Q2Q07770                                         Q2770330
          0             ACC                                              Q2770340
          BCD           1 X**I                                           Q2770350
```

```
PROC      STQ       ISAVE                                    Q2770360
+         QJP    P  *+1                                      Q2770370
          LQC       ISAVE                                    Q2770380
          STQ       I                                        Q2770390
+         CALL      Q1Q10770                                 Q2770400
          0         XTO2N                                    Q2770410
          LDA       I                                        Q2770420
          LRS       +1                                       Q2770430
          STA       I                                        Q2770440
+         CALL      Q1Q00770                                 Q2770450
          0         ONE                                      Q2770460
          QJP    P  *+2                                      Q2770470
+         CALL      Q1Q00770                                 Q2770480
          0         XTO2N                                    Q2770490
          ENI    1  0                                        Q2770500
+         CALL      Q1Q10770                                 Q2770510
          0         X                                        Q2770520
LOOPB     LDA       I                                        Q2770530
          AJP    Z  OUT                                      Q2770540
LOOPB1    LRS       +1                                       Q2770550
          QJP    M  LOOPC                                    Q2770560
          INI    1  +1                                       Q2770570
          SLJ       LOOPB1                                   Q2770580
LOOPC     STA       2                                        Q2770590
+         CALL      Q1Q00770                                 Q2770600
          0         XTO2N                                    Q2770610
LOOPC1    CALL      Q1Q04770                                 Q2770620
          0         XTO2N                                    Q2770630
          CALL      Q1Q10770                                 Q2770640
          0         XTO2N                                    Q2770650
          IJP    1  LOOPC1                                   Q2770660
+         CALL      Q1Q04770                                 Q2770670
          0         X                                        Q2770680
          CALL      Q1Q10770                                 Q2770690
          0         X                                        Q2770700
          SLJ       LOOPB                                    Q2770710
```

```
OUT      SSK        ISAVE                                    Q2770720
         SLJ        RELOAD                                   Q2770730
+        CALL       Q1Q00770                                 Q2770740
         0          ONE                                      Q2770750
         CALL       Q1Q05770                                 Q2770760
         0          X                                        Q2770770
EXIT     ENI     1  **                                       Q2770780
         SLJ        Q2Q07770                                 Q2770790
RELOAD   CALL       Q1Q00770                                 Q2770800
         0          X                                        Q2770810
         SLJ        EXIT                                     Q2770820
         END                                                 Q2770830
```

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1. ORIGINATING ACTIVITY (Corporate author) U. S. Naval Postgraduate School Monterey, California	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE
AUTOMATIC ERROR ANALYSIS IN FINITE DIGITAL COMPUTATIONS, USING RANGE ARITHMETIC

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Master's Thesis

5. AUTHOR(S) (Last name, first name, initial)
Monte Leon, Victor J., Lieutenant, United States Navy

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
May, 1966	81	5

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. AVAILABILITY/LIMITATION NOTICES

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY

13. ABSTRACT

The nature of generated machine error in finite digital calculations is discussed. The arithmetic of range numbers is developed, and examples are given demonstrating the use of range arithmetic as a tool for automatic error analysis. A computer program is developed, utilizing the TYPE OTHER feature of FORTRAN-63 in conjunction with the CDC-1604 digital computer, which enables the user to perform automatic error analysis during computation, and a number of programs are presented using this feature.

DD FORM 1 JAN 64 1473

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Automatic Error Analysis						
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